Why propositional quantification makes modal logics on trees robustly hard?

(joint paper with Stéphane Demri from CNRS)

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A concept of separation

A few examples:

- Quantified Modal logic $K$ [Fine 70]
- Quantified LTL [Stitt et al., 83]
- Logics of public announcement [Plaza, 89]
- Separation logic [Reynolds, 2002]
- Relation-changing Modal Logics [Aucher et al., 2009]
- Team modal logic [Müller 2014]
- Separating Modalities [Courtault et al., 2016]
- Modal separation logic [Fervari et al., 2018]
Proposalional quantification - a more general setting

- Separation $\approx$ colouring parts with different colours

\[
\exists.\phi \text{ def } \phi \text{ is satisfied after colouring } M
\]
Propositional quantification - a more general setting

- Separation $\approx$ colouring parts with different colours

- Propositional quantification

\[ \mathcal{M} \models \exists \circ. \varphi \overset{\text{def}}{=} \varphi \text{ is satisfied after colouring } \mathcal{M} \text{ with } \circ \]
Propositional Quantification = undecidability since 1970

- Modal logics: K, S4, GL.
- Temporal logics: QLTL, QCTL
- and even more...

The end of the story?

In this paper I shall present some of the results I have obtained on modal theories which contain quantifiers for propositions. The paper is in two parts: in the first part I consider theories whose non-quantificational subset is S5; in the second part I consider theories whose non-quantificational subset is weaker than or not contained in S5. Unless otherwise stated, each theory has the same language $L$. The elements of a nonempty set $V$ of propositional variables $p, q, \ldots$, the operators $\wedge$ (and), $\vee$ (or), $\neg$ (not) and $\Box$ (necessarily), the universal quantifier $\forall$, a propositional variable, and brackets ( and ). The formulas of $L$ are then defined in the usual way.
Propositional Quantification = undecidability since 1970

- **Modal logics**: $K$, $S4$, $GL$.
- **Temporal logics**: $QLTL$, $QCTL$
- and even more...

The end of the story?

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In this paper I shall present some of the results I have obtained on modal theories which contain quantifiers for propositions. The paper is in two parts: in the first part I consider theories whose non-quantificational contents is $S5$. In the second part I consider theories whose non-quantificational contents are weaker than or not contained in $S5$. Unless otherwise stated, each theory has the same language $L$. The syntax of $L$ involves a set $V$ of propositional variables $p_1, p_2, \ldots$, the operators $\lor$, $\land$, $\neg$ (or), $\neg\neg$ (not) and $\Box$ (necessarily), the universal quantifier $\forall$, propositional variables, and brackets ( and ). The formulas of $L$ are then defined in the usual way.

Not really. **Consider trees as models!**
Tree semantics: the cure for undecidability

The cure for undecidability

Instead of colouring models colour its tree unfolding!
Tree semantics: the cure for undecidability

Decidability on trees [Sistla et al, 87], [Laroussinie et al, 14]  
QCTL, QCTL* and QLTL on trees are TOWER-complete.
Tree semantics: the cure for undecidability

Decidability on trees [Sistla et al, 87], [Laroussinie et al, 14]

QCTL, QCTL* and QLTL on trees are TOWER-complete.

But what about standard modal logics? THIS TALK!
The main goal of this paper

What is the exact complexity of quantified MLs on trees?

TOWER-hardness for previous logics required until operator.

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So maybe modal logics are elementarily decidable?

The answer (unpleasant truth)
Quantified standard MLs on trees are TOWER-complete.
The main goal of this paper

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TOWER-hardness for previous logics required until operator.
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The answer (unpleasant truth)
Quantified standard MLs on trees are TOWER-complete.

We sketch the hardness proof for $\text{QCTL}_{\text{EX}} \approx \text{QK}$. 
Quantified Computation-Tree Logic with $\mathbf{X}$ only

- **atomic propositions:** $\bigcirc$, $\bigcirc$, ...

- **boolean combinators:** $\neg \varphi$, $\varphi \lor \psi$, $\varphi \land \psi$, ...

- **modalities:**
  
  - $\mathsf{EX} \varphi$
  
  - $\mathsf{AX} \varphi$

- **propositional quantifiers:**
  
  - $\exists \varphi$
  
  - $\forall \varphi$
Expressivity example: uniqueness

Non-uniqueness

\[ \exists \diamond (\text{EX}(\diamond \land \varphi) \land \text{EX}(\neg \diamond \land \varphi)) \]
Expressivity example: uniqueness

Non-uniqueness

\[ \exists \circ . \left( \text{EX}(\circ \land \varphi) \land \text{EX}(\neg \circ \land \varphi) \right) \]

Uniqueness

\[ \text{EX}(\varphi) \land \neg \exists \circ . \left( \text{EX}(\circ \land \varphi) \land \text{EX}(\neg \circ \land \varphi) \right) \]
A notion of local nominals

- Uniqueness expressible but only in the limited scope
- Local nominal = nominal but in limited scope
A notion of local nominals

- Uniqueness expressible but only in the limited scope
- Local nominal = nominal but in limited scope
- Useful operators: nom(x, lvl) (binder) and $\ominus_{x}^{\text{lvl}} \phi$ (at).

\[
\text{nom}(x, \text{lvl}) = \text{EX}_{=1}^{\text{lvl}}(x)
\]
\[
\ominus_{x}^{\text{lvl}} \phi = \text{EX}^{\text{lvl}}(x \land \phi)
\]

e.g. nom(x, 3)
Multiple nominals

Let \( \text{diff-nom}(x_1, \ldots, x_n, \text{lvl}) \) be

\[
\bigwedge_{i \in [1,n]} \text{nom}(x_i, k) \land \bigwedge_{i<j \in [1,n]} \neg \Theta_{x_i x_j}^{\text{lvl}} \land \\
\text{nom}(y, 1) \land \text{diff-nom}(x, z, 3)
\]
Enforcing exponential degree
An example of local nominals technique

- Label children with bits $P = \{p_0, p_1, \ldots, p_{n-1}\}$. 

There exists a node carrying zero.

$\exists x \neg p_0 \land \ldots \land \neg p_{n-1}$

There are no two nodes with the same number.

$\forall x, y \text{diff-nom}(x, y, 1) \rightarrow \neg \left( \bigwedge_{p \in P} p \leftrightarrow \bigwedge_{p \in P} \right)$
Enforcing exponential degree

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$$EX(\neg p_0 \land \neg p_1 \land \ldots \land \neg p_{n-1})$$
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There exists a node carrying zero.

$$\text{EX}(\neg p_0 \land \neg p_1 \ldots \land \neg p_{n-1})$$

There are no two nodes with the same number.

$$\forall x, y \, \text{diff-nom}(x, y, 1) \rightarrow \neg(\bigwedge_{p \in P} \mathcal{O}^1_x p \leftrightarrow \mathcal{O}^1_y p)$$
Successor relation (aka. adding plus one)

\[ \forall x \text{ nom}(x, 1) \land x \neq 2^n - 1 \rightarrow \exists y \text{ diff-nom}(x, y, 1) \land y = x + 1 \]

for all nodes \( x \) except the last

there is a successor \( y \)
How to express $y = x + 1$?

\[
\begin{array}{cccccccccccccccc}
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & x \\
+ & & & & & & & & & & & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & y = x + 1 \\
\end{array}
\]

\[
\begin{array}{c}
\left( \bigwedge_{i=0}^{n-1} \neg p_i \land \bigwedge_{j=0}^{i-1} p_j \right) \rightarrow \\
\left( \bigwedge_{j=0}^{i-1} \neg p_j \land p_i \right) \land \\
\bigwedge_{j=i+1}^{n-1} @^1_x(p_j) \leftrightarrow @^1_y(p_j) \\\n\end{array}
\]

look for the first zero bit
reset previous bits, set $p_i$
rewrite other bits
How to prove TOWER–hardness? Part I: k–Tillings

\[ \exp(1, n) = 2^n, \quad \exp(k + 1, n) = 2^{\exp(k, n)} \]

**Constraints**

- Finite set of puzzles
- Horizontal and vertical constraints
- Goal: Tile a board of the size \( \exp(k, n) \times \exp(k, n) \)

**Rules**

- Finite set of puzzles
- Horizontal and vertical constraints
- Goal: Tile a board of the size \( \exp(k, n) \times \exp(k, n) \)
How to prove TOWER-hardness? Part II: Huge degree

Type $k$

Type $(k-1)$

Type $(k-2)$

Type 0

$p_{n-1} = \cdots = p_1 = \bot, p_0 = \top$
Enforcing doubly-exponential degree

Number of a node $\sim$ encoded on val predicates
Enforcing doubly-exponential degree

Number of a node $\sim$ encoded on val predicates

- There exists a node carrying zero. $\text{EX(AX(\neg\text{val}))}$
Enforcing doubly-exponential degree

Number of a node $\sim$ encoded on val predicates

- There exists a node carrying zero. $\text{EX}(\text{AX}(\neg\text{val}))$
- There are no two nodes with the same number. ???
- Every node has successor. ???
There are no two nodes with the same number.
There are no two nodes with the same number.

\[ \forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg \text{equalNum}(x, y) \]
There are no two nodes with the same number.

\[
\neg \text{equalNum}(x, y) \overset{\text{def}}{=} \\
\exists a \exists b \ (\text{diff-nom}(a, b, 2) \land \Diamond^1_x (\text{EX } a) \land \Diamond^1_y (\text{EX } b)) \land \\
\text{equalNum}(a, b) \land \neg (\Diamond^2_a (\text{val}) \land \Diamond^2_b (\text{val}))
\]
There are no two nodes with the same number.

\[
\neg \text{equalNum}(x, y) \overset{\text{def}}{=} \\
\exists a \exists b \left( \text{diff-nom}(a, b, 2) \land \varpi^1_x (\text{EX } a) \land \varpi^1_y (\text{EX } b) \right) \land \\
\text{equalNum}(a, b) \land \neg (\varpi^2_a (\text{val}) \land \varpi^2_b (\text{val}))
\]
There are no two nodes with the same number.

\[
\neg \text{equalNum}(x, y) \overset{\text{def}}{=} \\
\exists a \exists b \, (\text{diff-nom}(a, b, 2) \land \mathcal{G}^1_x(\text{EX } a) \land \mathcal{G}^1_y(\text{EX } b)) \land \\
\text{equalNum}(a, b) \land \neg (\mathcal{G}^2_a(\text{val}) \land \mathcal{G}^2_b(\text{val}))
\]

What about successor relation?
More general way of adding plus one
an abstraction of the previous technique

\[
\begin{array}{c}
1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \\
+ \ \\
\hline
1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0
\end{array}
\]

\[y = x + 1\]

A nice abstraction:

<table>
<thead>
<tr>
<th></th>
<th>left, to be rewritten</th>
<th>selector = 0</th>
<th>right = 111\ldots1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x+1</td>
<td></td>
<td>selector = 1</td>
<td>right = 000\ldots0</td>
</tr>
</tbody>
</table>
Summing it up

A part of the main result

\[ \text{QCTL}_{\text{EX}} \text{ on trees is } k\text{–NExpTime–hard for each } k \in \mathbb{N}. \]
Summing it up

A part of the main result

QCTL\textsubscript{EX} on trees is k–NExpTime–hard for each k \in \mathbb{N}.

Reduction was uniform, so QCTL\textsubscript{EX} is TOWER–hard.

The upper bound from MSO on trees.
Summing it up

**A part of the main result**

QCTL\textsubscript{EX} on trees is $k$–NExpTime–hard for each $k \in \mathbb{N}$.

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**The main result**

Over trees QCTL\textsubscript{EX} is TOWER–complete.
Summing it up

A part of the main result

\[ \text{QCTL}_\text{EX} \text{ on trees is } k\text{-NExpTime–hard for each } k \in \mathbb{N}. \]

Reduction was uniform, so \text{QCTL}_\text{EX} \text{ is TOWER–hard.}

The upper bound from MSO on trees.

The main result

Over trees \text{QCTL}_\text{EX} \text{ is TOWER–complete.}

Main ingredient = Huge degree.

What happens when degree is bounded?
Trees with bounded degree

**Bounded degree trees**

$\text{QCTL}^\text{EX}_\text{EX}$ is $\text{AExpPol}$-complete on trees with bounded degree.

AExpPol = alternating exp time with poly alternations

Main ingredients:
Trees with bounded degree

Bounded degree trees

\( \text{QCTL}_{\text{EX}} \) is \( \text{AExpPol-complete} \) on trees with bounded degree.

\( \text{AExpPol} = \text{alternating exp time with poly alternations} \)

Main ingredients:

- **Upper bound** = \( \text{exp models} + \text{model checking algorithm} \)
- **Lower bound** = \( \text{exp multi-tilings} \) [Bozzelli et al, 2018]
Conclusions

Our results (arbitrary trees)
Quantified K (aka. $\text{QCTL}_{\text{EX}}$) on trees is TOWER-complete. Hardness applies also to $\text{GL}$, $\text{S4}$, $\text{K4}$, $\text{KD}$, $\text{QCTL}_{\text{EF}}$ on trees.

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Quantified K (aka. \( \text{QCTL}_{\text{EX}} \)) on trees is TOWER-complete. Hardness applies also to GL, S4, K4, KD, \( \text{QCTL}_{\text{EF}} \) on trees.

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\( \text{QCTL}_{\text{EX}} \) is \( \text{AExpPol} \)-complete on trees with bounded degree.

Open problems:
- Expressive power of \( \text{QCTL}_{\text{EX}} \)?
- Nice elementary fragments?
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Quantified K (aka. QCTL$_{EX}$) on trees is TOWER-complete. Hardness applies also to GL, S4, K4, KD, QCTL$_{EF}$ on trees.

Our results (bounded degree trees)

QCTL$_{EX}$ is AExpPol-complete on trees with bounded degree.

Open problems:

- Expressive power of QCTL$_{EX}$?
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