On Classical Decidable Logics
Extended with Percentage Quantifiers and Arithmetics

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Motivating example: election knowledge-bases
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We formalise the concept of a **winner** with:

- Must-have: inverses of relations + percentages
- No decidable logic available

**Related work**

1. Presburger Modal Logic \[\text{Demri&Lugiez'2010}\]
2. ALC with Expressive Cardinality Constraints (ALCSCC) \[\text{Baader'2017}\]
3. Coalgebraic Modal Logics [e.g. works of Schröder, Pattinson, Kupke and many more]
4. ALCISCC++ [Baader et. al'2020]

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Classical Logics with Percentages and/or Arithmetics

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Motivating example: election knowledge-bases

We formalise the concept of a **winner** with:

\[ \text{Citizen} \sqsubseteq \exists \text{votedFor} \]

**Must-have:** inverses of relations + percentages

**No decidable logic available**

**Related work**
1. Presburger Modal Logic \([\text{Demri} \& \text{Lugiez}'2010]\)
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4. \(\text{ALCISCC}^{++}\) \([\text{Baader et. al}'2020]\)
Motivating example: election knowledge-bases

Citizen(Bart)

votedFor(Bart, XYZ)
Motivating example: election knowledge-bases

Database

Citizen(Bart)
votedFor(Bart, XYZ)

Knowledge

We formalise the concept of a winner with:

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Classical Logics with Percentages and/or Arithmetics
Motivating example: election knowledge-bases

\[\text{Citizen}(\text{Bart})\]
\[\forall \text{votedFor}(\text{Bart}, \text{XYZ})\]

\[\exists \text{votedFor}. \text{Citizen}\]
Motivating example: election knowledge-bases

We formalise the concept of a winner with:

\[\text{Citizen}(\text{Bart}) \land \text{votedFor}(\text{Bart}, \text{XYZ}) \implies \exists \text{votedFor}. \text{Citizen}\]
Motivating example: election knowledge-bases

We formalise the concept of a winner with:

\[\text{winner} \equiv \text{Citizen} \cap (\geq 50\%) \text{votedFor}^{-} \cdot \text{Citizen}\]

Citizen(Bart)
votedFor(Bart, XYZ)

Citizen \subseteq \exists \text{votedFor} \cdot \text{Citizen}
Motivating example: election knowledge-bases

We formalise the concept of a winner with:

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Must-have: inverses of relations + percentages

\(\text{Citizen} \sqsubseteq \exists \text{votedFor}. \text{Citizen}\)
Motivating example: election knowledge-bases

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Must-have: inverses of relations + percentages

No decidable logic available 🙁

\[
\text{Citizen} \sqsubseteq \exists \text{votedFor}.\text{Citizen}
\]
Motivating example: election knowledge-bases

We formalise the concept of a *winner* with:

\[
\text{winner} \equiv \text{Citizen} \cap (>50\%) \text{votedFor}^{-}\cdot\text{Citizen}
\]

We need inverses of relations + percentages.

No decidable logic available 😞

\[
\text{Citizen} \sqsubseteq \exists \text{votedFor} \cdot \text{Citizen}
\]

Related work

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Related work

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No decidable logic available 😞

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Related work

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2. \(ALC\) with Expressive Cardinality Constraints (\(ALCSCC\)) [Baader’2017] 😞
3. Coalgebraic Modal Logics [e.g. works of Schröder, Pattinson, Kupke and many more] 😊
4. \(ALCISCC^{++}\) [Baader et. al’2020] 😞 or \(FO^2\) with Härtig quantifier [Grädel et al.’1999] 😞
We focus on classical decidable fragments: $\text{FO}^2$ and $\text{GF}$
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1. $\text{FO}^2$ is the fragment of $\text{FO}$, in which we can only use the variables $x$ and $y$
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   - not so trivial as we can reuse variables, e.g. \( \forall x \exists y (E(x, y) \land \exists x (E(y, x) \land \exists y E(x, y))) \)

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Classical Logics with Percentages and/or Arithmetics

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We focus on classical decidable fragments: FO² and GF

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2. The guarded fragment $\text{GF}$ of FO is obtained by relativising quantifiers by atoms.
   - $\exists \bar{y} \alpha(\bar{x}, \bar{y}) \land \phi(\bar{x}, \bar{y}), \forall \bar{y} \alpha(\bar{x}, \bar{y}) \rightarrow \phi(\bar{x}, \bar{y})$ – guard must cover free variables of $\phi$. 
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2. The guarded fragment \( \text{GF} \) of \( \text{FO} \) is obtained by relativising quantifiers by atoms.
   - \( \exists \bar{y} \alpha(\bar{x}, \bar{y}) \land \varphi(\bar{x}, \bar{y}), \forall \bar{y} \alpha(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}) \) – guard must cover free variables of \( \varphi \).
   - also has FMP, 2ExpTime-complete SAT [Grädel 1999]
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   - $\exists \vec{y} \alpha(\vec{x}, \vec{y}) \land \varphi(\vec{x}, \vec{y}), \forall \vec{y} \alpha(\vec{x}, \vec{y}) \rightarrow \varphi(\vec{x}, \vec{y})$ – guard must cover free variables of $\varphi$.
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Example: Some artist admires only beekeepers

$$\exists x \text{artst}(x) \land \forall y (\text{adm}(x, y) \rightarrow \text{bkpr}(y))$$
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Example: Some artist admires only beekeepers

$$\exists x \text{artst}(x) \land \forall y (\text{adm}(x, y) \rightarrow \text{bkpr}(y))$$

Coexample: Every artist admires every beekeeper

$$\forall x (\text{artst}(x) \rightarrow \forall y (\text{bkpr}(y) \rightarrow \text{adm}(x, y)))$$
We focus on classical decidable fragments: \( \text{FO}^2 \) and \( \text{GF} \)

1. \( \text{FO}^2 \) is the fragment of \( \text{FO} \), in which we can only use the variables \( x \) and \( y \)
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2. The guarded fragment \( \text{GF} \) of \( \text{FO} \) is obtained by relativising quantifiers by atoms.
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\[ \exists x \ \text{artst}(x) \land \forall y \ (\text{adm}(x, y) \rightarrow \text{bkpr}(y)) \]

Coexample: Every artist admires every beekeeper

\[ \forall x \ (\text{artst}(x) \rightarrow \forall y \ (\text{bkpr}(y) \rightarrow \text{adm}(x, y))) \]

Both \( \text{FO}^2 \) and \( \text{GF} \) capture \( \mathcal{ALCI} \) but cannot express percentages.
We focus on classical decidable fragments: FO² and GF

1. FO² is the fragment of FO, in which we can only use the variables x and y
   - has FMP and Exp-size model property [Grädel&Kolaitis&Vardi 1997], NExpTime-complete SAT
   - not so trivial as we can reuse variables, e.g. \( \forall x \exists y (E(x, y) \land \exists x (E(y, x) \land \exists y E(x, y))) \)

2. The guarded fragment GF of FO is obtained by relativising quantifiers by atoms.
   - \( \exists \bar{y} \alpha(\bar{x}, \bar{y}) \land \varphi(\bar{x}, \bar{y}), \forall \bar{y} \alpha(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}) \) – guard must cover free variables of \( \varphi \).
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\[ \exists x \; \text{artst}(x) \land \forall y \; (\text{adm}(x, y) \rightarrow \text{bkpr}(y)) \]

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Both FO² and GF capture ALCI but cannot express percentages.

So let’s add them! Why not?
Percentage quantifiers and the two semantics

Our positive results hold even for Presburger's arithmetic \(\mathsf{FO}^{+}\) constraints on successors.

1. **Global percentage quantifiers**
   
   \[ \exists \leq k \% x. \varphi, \exists > k \% x. \varphi, \exists < k \% x. \varphi \]

   count globally

   \[ A, \vec{a} \models \exists \leq k \% x. \varphi \iff \exists d \in A : A, \vec{a} \models \varphi(d) \leq \frac{k}{100} \]

2. **Local percentage quantifiers**

   \[ \exists \leq k \% R x. \varphi, \exists > k \% R x. \varphi, \exists < k \% R x. \varphi \]

   count successors

   \[ A, \vec{a} \models \exists \leq k \% R x. \varphi \iff \exists (\vec{a}, d) \in R : A \models \varphi(d) \leq \frac{k}{100} \]

Our results

- \(\mathsf{FINSAT}\) for \(\mathsf{FO}^{2\%}\) is undecidable under any semantics.
- \(\mathsf{FINSAT}\) for \(\mathsf{GF}^{+\%}\) is undecidable under any semantics. Thus we meet in the middle and focus on \(\mathsf{GF}^{2\%} := \mathsf{FO}^{2} \setminus \mathsf{GF}\).
- \(\mathsf{FINSAT}\) for \(\mathsf{GF}^{2\%}\) global % is undecidable.
- \(\mathsf{FINBranchSAT}\) for \(\mathsf{GF}^{2\%}\) local % is \(\text{ExpTime}\)-complete and \(\text{CQ}\) querying is \(\text{2ExpTime}\)-complete.
- \(\mathsf{FINSAT}\) for \(\mathsf{GF}^{2\%}\) local % is in \(\text{3NExpTime}\) + \(\text{CQ}\) querying in \(\text{4NExpTime}\).
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists = k \% x. \varphi$, $\exists > k \% x. \varphi$, $\exists < k \% x. \varphi$ count globally
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists^{=k}\!\!\!\!\!\!\!\!\!\!\!\!\text{x} . \varphi$, $\exists^{>k}\!\!\!\!\!\!\!\!\!\!\!\!\text{x} . \varphi$, $\exists^{<k}\!\!\!\!\!\!\!\!\!\!\!\!\text{x} . \varphi$ count globally

$$\mathcal{A}, \overrightarrow{a} \models \exists^{=k}\!\!\!\!\!\!\!\!\!\!\!\!\text{x} . \varphi \text{ iff } \forall d \in A : \mathcal{A}, \overrightarrow{a} \models \varphi(d) = \frac{k}{100} |A|$$
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists=^k x.\varphi$, $\exists>^k x.\varphi$, $\exists<^k x.\varphi$ count globally

   $\mathcal{A}, \vec{a} \models \exists=^k x.\varphi$ iff $|d \in \mathcal{A} : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |\mathcal{A}|$

2. Local percentage quantifiers $\exists_R=^k x.\varphi$, $\exists_R>^k x.\varphi$, $\exists_R<^k x.\varphi$ count successors

   $\mathcal{A}, \vec{a} \models \exists_R=^k x.\varphi$ iff $|d \in \mathcal{A} : (\vec{a}, d) \in R| = \frac{k}{100} |\mathcal{A}|$
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists^{\geq k\%}x.\varphi$, $\exists^{> k\%}x.\varphi$, $\exists^{< k\%}x.\varphi$ count globally

   $\mathcal{A}, \vec{a} \models \exists^{= k\%}x.\varphi$ iff $|d \in \mathcal{A} : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |\mathcal{A}|

2. Local percentage quantifiers $\exists_{R}^{= k\%}x.\varphi$, $\exists_{R}^{> k\%}x.\varphi$, $\exists_{R}^{< k\%}x.\varphi$ count successors

   $\mathcal{A}, \vec{a} \models \exists_{R}^{= k\%}x.\varphi$ iff $|d \in \mathcal{A} : (\vec{a}, d) \in R_{\mathcal{A}}$ and $\mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in \mathcal{A} : (\vec{a}, d) \in R_{\mathcal{A}}|$
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists^{=}k%x, \exists^{>}k%x, \exists^{<}k%x$ count globally

   $\mathcal{A}, \vec{a} \models \exists^{=}k%x.\varphi \iff |d \in A : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|A|$

2. Local percentage quantifiers $\exists^{=}_R k%x, \exists^{>}_R k%x, \exists^{<}_R k%x$ count successors

   $\mathcal{A}, \vec{a} \models \exists^{=}_R k%x.\varphi \iff |d \in A : (\vec{a}, d) \in R^a \text{ and } \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|d \in A : (\vec{a}, d) \in R^a|$

Note that global (resp. local) % make sense only over finite (resp. finite-branching) structures.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists = k \% x . \varphi$, $\exists > k \% x . \varphi$, $\exists < k \% x . \varphi$ count globally

    $\mathcal{A}, \vec{a} \models \exists = k \% x . \varphi$ iff $|d \in A : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |A|$

2. Local percentage quantifiers $\exists_R = k \% x . \varphi$, $\exists_R > k \% x . \varphi$, $\exists_R < k \% x . \varphi$ count successors

    $\mathcal{A}, \vec{a} \models \exists_R = k \% x . \varphi$ iff $|d \in A : (\vec{a}, d) \in R^A$ and $\mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in A : (\vec{a}, d) \in R^A|$

Note that global (resp. local) % make sense only over finite (resp. finite-branching) structures.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists = k \% x . \varphi$, $\exists > k \% x . \varphi$, $\exists < k \% x . \varphi$ count globally
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2. Local percentage quantifiers $\exists = k \% R x . \varphi$, $\exists > k \% R x . \varphi$, $\exists < k \% R x . \varphi$ count successors
   $\mathcal{A}, \vec{a} \models \exists = k \% R x . \varphi$ iff $|d \in A : (\vec{a}, d) \in R^\mathcal{A}$ and $\mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in A : (\vec{a}, d) \in R^\mathcal{A}|$

Note that global (resp. local) % make sense only over finite (resp. finite-branching) structures.

Our results

• FINSAT for FOL $^2 + \%$ is undecidable under any semantics.
• FINSAT for GF $^2 + \%$ is undecidable under any semantics.
Thus we meet in the middle and focus on GF $^2 := \text{FO}^2 \setminus \text{GF}$.
• FINSAT for GF $^2 + \%$ global is undecidable.
• FINBranchSAT for GF $^2 + \%$ local is ExpTime-complete and CQ querying is 2ExpTime-complete.
• FINSAT for GF $^2 + \%$ local is in 3NExpTime and CQ querying in 4NExpTime.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists^= k^\% x. \varphi$, $\exists^> k^\% x. \varphi$, $\exists^< k^\% x. \varphi$ count globally

$$\mathcal{A}, \vec{a} \models \exists^= k^\% x. \varphi \iff |d \in A : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |A|$$

2. Local percentage quantifiers $\exists^= R^k x. \varphi$, $\exists^> R^k x. \varphi$, $\exists^< R^k x. \varphi$ count successors

$$\mathcal{A}, \vec{a} \models \exists^= R^k x. \varphi \iff |d \in A : (\vec{a}, d) \in R^A \text{ and } \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in A : (\vec{a}, d) \in R^A|$$

Note that global (resp. local) $\%$ make sense only over finite (resp. finite-branching) structures.

Our results

- FINSAT for FO$^2 + \%$ is undecidable under any semantics.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers \( \exists = k \% x . \varphi, \exists > k \% x . \varphi, \exists < k \% x . \varphi \) count globally
   \[ \mathcal{A}, \vec{a} \models \exists = k \% x . \varphi \text{ iff } |d \in A : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |A| \]

2. Local percentage quantifiers \( \exists_R = k \% x . \varphi, \exists_R > k \% x . \varphi, \exists_R < k \% x . \varphi \) count successors
   \[ \mathcal{A}, \vec{a} \models \exists_R = k \% x . \varphi \text{ iff } |d \in A : (\vec{a}, d) \in R^\mathcal{A} \text{ and } \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in A : (\vec{a}, d) \in R^\mathcal{A}| \]

Note that global (resp. local) \( \% \) make sense only over finite (resp. finite-branching) structures.

Our results

- FINSAT for FO\(^2\) + \% is undecidable under any semantics.
- FINSAT for GF + \% is undecidable under any semantics.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists = k% x . \varphi$, $\exists > k% x . \varphi$, $\exists < k% x . \varphi$ count globally

   $\mathcal{A}, \vec{a} \models \exists = k% x . \varphi$ iff $|d \in A : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |A|$

2. Local percentage quantifiers $\exists_R = k% x . \varphi$, $\exists_R > k% x . \varphi$, $\exists_R < k% x . \varphi$ count successors

   $\mathcal{A}, \vec{a} \models \exists_R = k% x . \varphi$ iff $|d \in A : (\vec{a}, d) \in R^\mathcal{A}$ and $\mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |d \in A : (\vec{a}, d) \in R^\mathcal{A}|$

Note that global (resp. local) $\%$ make sense only over finite (resp. finite-branching) structures.

Our results

- FINSAT for $\text{FO}^2 + \%$ is undecidable under any semantics.
- FINSAT for $\text{GF}^2 + \%$ is undecidable under any semantics.

Thus we meet in the middle and focus on $\text{GF}^2 := \text{FO}^2 \cap \text{GF}$. 
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists^=k%x.\varphi$, $\exists^>k%x.\varphi$, $\exists^<k%x.\varphi$ count globally
   \[ \mathfrak{A}, \vec{a} \models \exists^=k%x.\varphi \iff |d \in A : \mathfrak{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|A| \]

2. Local percentage quantifiers $\exists^=R^k%x.\varphi$, $\exists^>R^k%x.\varphi$, $\exists^<R^k%x.\varphi$ count successors
   \[ \mathfrak{A}, \vec{a} \models \exists^=R^k%x.\varphi \iff |d \in A : (\vec{a}, d) \in R^A \text{ and } \mathfrak{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|d \in A : (\vec{a}, d) \in R^A| \]

   Note that global (resp. local) % make sense only over finite (resp. finite-branching) structures.

Our results

- FINSAT for $FO^2 + %$ is undecidable under any semantics.
- FINSAT for $GF + %$ is undecidable under any semantics.
  Thus we meet in the middle and focus on $GF^2 := FO^2 \cap GF$.
- FINSAT for $GF^2 + \text{global } %$ is undecidable.
Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists=k\% x. \varphi$, $\exists> k\% x. \varphi$, $\exists< k\% x. \varphi$ count globally

\[ \mathcal{A}, \vec{a} \models \exists=k\% x. \varphi \text{ iff } |d \in \mathcal{A} : \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |\mathcal{A}| \]

2. Local percentage quantifiers $\exists_R=k\% x. \varphi$, $\exists_R> k\% x. \varphi$, $\exists_R< k\% x. \varphi$ count successors

\[ \mathcal{A}, \vec{a} \models \exists_R=k\% x. \varphi \text{ iff } |d \in \mathcal{A} : (\vec{a}, d) \in R^\mathcal{A} \text{ and } \mathcal{A}, \vec{a} \models \varphi(d)| = \frac{k}{100} |\mathcal{A}| \]

Note that global (resp. local) $\%$ make sense only over finite (resp. finite-branching) structures.

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- FINSAT for $\text{GF}^2 + \text{global } \%$ is undecidable.
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Percentage quantifiers and the two semantics

1. Global percentage quantifiers $\exists=k\% x. \phi$, $\exists>k\% x. \phi$, $\exists<k\% x. \phi$ count globally

   \[ \mathcal{A}, \vec{a} \models \exists=k\% x. \phi \text{ iff } |d \in A : \mathcal{A}, \vec{a} \models \phi(d)| = \frac{k}{100}|A| \]

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- FINSAT for $\text{GF}^2 + \text{local} \%$ is in $3\text{NExpTime} + \text{CQ querying in } 4\text{NExpTime}$. 

Percentage quantifiers and the two semantics

1. Global percentage quantifiers \( \exists = k \% x. \varphi \), \( \exists > k \% x. \varphi \), \( \exists < k \% x. \varphi \) count globally

\[ \mathfrak{A}, \vec{a} \models \exists = k \% x. \varphi \iff |d \in A : \mathfrak{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|A| \]

2. Local percentage quantifiers \( \exists_R = k \% x. \varphi \), \( \exists_R > k \% x. \varphi \), \( \exists_R < k \% x. \varphi \) count successors

\[ \mathfrak{A}, \vec{a} \models \exists_R = k \% x. \varphi \iff |d \in A : (\vec{a}, d) \in R^{\mathfrak{A}} \text{ and } \mathfrak{A}, \vec{a} \models \varphi(d)| = \frac{k}{100}|d \in A : (\vec{a}, d) \in R^{\mathfrak{A}}| \]

Note that global (resp. local) % make sense only over finite (resp. finite-branching) structures.

Our results

- FINSAT for FO^2 + % is undecidable under any semantics.
- FINSAT for GF + % is undecidable under any semantics.

Thus we meet in the middle and focus on GF^2 := FO^2 \cap GF.

- FINSAT for GF^2 + global % is undecidable.
- FINBranchSAT for GF^2 + local % is \texttt{ExpTime}-complete and CQ querying is \texttt{2ExpTime}-complete.
- FINSAT for GF^2 + local % is in \texttt{3NExpTime} + CQ querying in \texttt{4NExpTime}.

Our positive results hold even for Presburger’s arithmetic (FO[+]) constraints on successors.
Overview of the proofs (undecidability)

1. Undecidability of $\text{FO} + \%$ and $\text{GF}_2 + \text{global} \%$.
   - We can axiomatise universal roles: $\forall x \forall y \ R(x, y)$.
   - So we can put dummy guards everywhere and the semantics of $\%$ doesn't matter.
   - Reduction from the Hilbert’s 10th problem. (Similarly to [Baader & B. & Rudolph, ECAI’20].)

2. Undecidability of $\text{GF}_3$ with local $\%$.
   - $\text{GF}_3 + \text{functional role}$ is undecidable [Grädel & Otto & Rosen’1999].
   - We show how to enforce functionality with $\%$.
Overview of the proofs (undecidability)

1. Undecidability of FO$^2$ + % and GF$^2$ + global %.

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2. Undecidability of GF with local %.

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Overview of the proofs (undecidability)

1. Undecidability of FO² + % and GF² + global %.
   - We can axiomatise universal roles: $\forall x \forall y \ R(x, y)$
   - So we can put dummy guards everywhere and the semantics of % doesn’t matter.
   - Reduction from the Hilbert’s 10th problem. (Similarly to [Baader&B.&Rudolph, ECAI’20])

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1. Undecidability of FO\(^2\) + % and GF\(^2\) + global %.
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Overview of the proofs (decidability)

1. ExpTime-completeness of FinBranchSAT of GF²

- If there is a model then there is an infinite tree-like one with exponential branching.
- APspace procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of GF²

- Translate into C², i.e. the FO² with counting (NExpTime-compl. [Pratt-Hartmann'2005])
- Rewrite a formula into some simplified form (reduce nesting depth to ≤ 2).
- Express local neighbourhood with Presburger formula over “types”.
- Such (semi-)linear constraints can be computed (with huge blow-up) and described in C².

3. Decidability of CQ query entailment

- Exponential reduction to satisfiability, based on “pumping” from [Baader&B.&Rudolph, DL’2019].
Overview of the proofs (decidability)

1. \textsc{ExpTime}-completeness of FinBranchSAT of GF^2 + local %.

- If there is a model then there is an infinite tree-like one with exponential branching.
- \textsc{APspace} procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of GF^2 + local %.

- Translate into C^2, i.e. the FO^2 with counting (\textsc{NExpTime}-compl. [Pratt-Hartmann'2005])
- Rewrite a formula into some simplified form (reduce nesting depth to \(\leq 2\)).
- Express local neighbourhood with Presburger formula over "types".
- Such (semi-)linear constraints can be computed (with huge blow-up) and described in C^2.

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1. \textsc{ExpTime}-completeness of FinBranchSAT of GF$^2$ + local \%.
   - If there is a model then there is an \textit{infinite} tree-like one with exponential branching.
Overview of the proofs (decidability)

1. **ExpTime**-completeness of FinBranchSAT of GF$^2 +$ local %.
   - If there is a model then there is an *infinite* tree-like one with exponential branching.
   - **APspace** procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of GF$^2 +$ local %.
   - Translate into C$^2$, i.e. the FO$^2$ with counting ([Pratt-Hartmann'2005](#)).
   - Rewrite a formula into some simplified form (reduce nesting depth to $\leq 2$).
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Overview of the proofs (decidability)

1. **ExpTime**-completeness of FinBranchSAT of GF$^2 + \text{local %}$.  
   - If there is a model then there is an *infinite* tree-like one with exponential branching.  
   - **APspace** procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of GF$^2 + \text{local %}$.  
   - Translate into C$^2$, i.e. the FO$^2$ with counting ([NExpTime compl. \cite{PrattHartmann2005}]).  
   - Rewrite a formula into some simplified form (reduce nesting depth to $\leq 2$).  
   - Express local neighbourhood with Presburger formula over “types”.  
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   - Exponential reduction to satisfiability, based on “pumping” from \cite{BaaderBR2019}.
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1. **ExpTime**-completeness of FinBranchSAT of $GF^2 + \text{local } \%$.
   - If there is a model then there is an *infinite* tree-like one with exponential branching.
   - **APspace** procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of $GF^2 + \text{local } \%$.
Overview of the proofs (decidability)

1. **ExpTime-completeness of FinBranchSAT of GF$^2$ + local %.

- If there is a model then there is an *infinite* tree-like one with exponential branching.
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2. Decidability of FINSAT of GF$^2$ + local %.

- Translate into $C^2$, i.e. the FO$^2$ with counting ($N\text{ExpTime}$-compl. [Pratt-Hartmann’2005])

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Bartosz “Bart” Bednarczyk

Classical Logics with Percentages and/or Arithmetics
Overview of the proofs (decidability)

1. **ExpTime**-completeness of FinBranchSAT of GF² + local %.
   - If there is a model then there is an *infinite* tree-like one with exponential branching.
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2. Decidability of FINSAT of GF² + local %.
   - Translate into C², i.e. the FO² with counting (**NExpTime**-compl. [Pratt-Hartmann’2005])
   - Rewrite a formula into some simplified form (reduce nesting depth to ≤ 2).
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1. ExpTime-completeness of FinBranchSAT of GF$^2$ + local %.
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Overview of the proofs (decidability)

1. \textsc{ExpTime}-completeness of \textsc{FinBranchSAT} of GF$^2 + \text{local} \%$.
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Overview of the proofs (decidability)

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   - Translate into C$^2$, i.e. the FO$^2$ with counting (NExpTime-compl. [Pratt-Hartmann’2005])
   - Rewrite a formula into some simplified form (reduce nesting depth to $\leq 2$).
   - Express local neighbourhood with Presburger formula over “types”.
   - Such (semi-)linear constraints can be computed (with huge blow-up) and described in C$^2$. 
Overview of the proofs (decidability)

1. EXPTime-completeness of FinBranchSAT of GF$^2$ + local %.
   - If there is a model then there is an *infinite* tree-like one with exponential branching.
   - APSPACE procedure: guess the model step by step (a la tableaux) and stop after exp-many steps.

2. Decidability of FINSAT of GF$^2$ + local %.
   - Translate into C$^2$, i.e. the FO$^2$ with counting (NEEXPTime-compl. [Pratt-Hartmann’2005])
   - Rewrite a formula into some simplified form (reduce nesting depth to $\leq 2$).
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   - Express local neighbourhood with Presburger formula over “types”.
   

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3. Decidability of CQ query entailment
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1. **ExpTime**-completeness of FinBranchSAT of $GF^2 + \text{local}\%$.
   - If there is a model then there is an *infinite* tree-like one with exponential branching.
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   - Translate into $C^2$, i.e. the $FO^2$ with counting ($NE\text{xpTime}$-compl. [Pratt-Hartmann'2005])
   - Rewrite a formula into some simplified form (reduce nesting depth to $\leq 2$).
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3. Decidability of CQ query entailment
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A glance at the undecidability proof of $FO^2 + \text{global } \%$: Two tricks
A glance at the undecidability proof of $FO^2 + \text{global \%}$: Two tricks

1. Call $\mathcal{A}$ (Half, R, J)-separated iff

\[
F \mid A = \forall x (\exists y F(x, y)) \rightarrow (\exists y = 50\% y (\text{Half}(y) \land x \neq y) \lor F(x, y))
\]
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Two tricks

1. Call $\mathfrak{A}$ $(\text{Half}, R, J)$-separated iff
   - The symbol $\text{Half}$ labels exactly half of domain elements, and
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Two tricks

1. Call $\mathcal{A}$ (Half, R, J)-separated iff
   - The symbol Half labels exactly half of domain elements, and
   - the elements labelled with $R$ and $J$ are disjoint and in different halves of $\mathcal{A}$.
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1. Call $\mathcal{A}$ $(\text{Half}, R, J)$-separated iff

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- the elements labelled with $R$ and $J$ are disjoint and in different halves of $\mathcal{A}$.

Over $(\text{Half}, R, J)$-separated $\mathcal{A}$ we can express $|R^{2\%}| = |J^{2\%}|$. 

[Diagram showing $\text{Half}$ and $\neg\text{Half}$ with $R$ and $J$]
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Two tricks

1. Call $\mathcal{A}$ (Half, $R$, $J$)-separated iff
   - The symbol $\text{Half}$ labels exactly half of domain elements, and
   - the elements labelled with $R$ and $J$ are disjoint and in different halves of $\mathcal{A}$.

Over (Half, $R$, $J$)-separated $\mathcal{A}$ we can express $|R^{2\mathcal{A}}| = |J^{2\mathcal{A}}|$.

\[
\models \varphi_{eq}(\text{Half, } R, J) := \exists^{=50\%} x \ (\text{Half}(x) \land \neg R(x)) \lor J(x)
\]
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Two tricks

1. Call $\mathcal{A}$ ($\text{Half}, R, J$)-separated iff

- The symbol $\text{Half}$ labels exactly half of domain elements, and
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Over ($\text{Half}, R, J$)-separated $\mathcal{A}$ we can express $|R^{|A}| = |J^{|A}|$.

$\models \varphi_{eq}(\text{Half}, R, J) := \exists^{=50\%} x (\text{Half}(x) \land \neg R(x)) \lor J(x)$

Over ($\text{Half}, R, J$)-separated $\mathcal{A}$ we can express that $F : R^{|A|} \to J^{|A|}$ is functional.
A glance at the undecidability proof of $\text{FO}^2 + \text{global }\%$: Two tricks

1. Call $\mathcal{A}$ $(\text{Half}, R, J)$-separated iff

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Over $(\text{Half}, R, J)$-separated $\mathcal{A}$ we can express $|R^{\mathcal{A}}| = |J^{\mathcal{A}}|$.

\[\mathcal{A} \models \varphi_{eq}(\text{Half}, R, J) := \exists^{50\%} x \left( \text{Half}(x) \land \neg R(x) \right) \lor J(x)\]

Over $(\text{Half}, R, J)$-separated $\mathcal{A}$ we can express that $F : R^{\mathcal{A}} \rightarrow J^{\mathcal{A}}$ is functional.
A glance at the undecidability proof of $\text{FO}_2^2 + \text{global } \%$: Two tricks

1. Call $\mathcal{A}$ (Half, R, J)-separated iff
   - The symbol Half labels exactly half of domain elements, and
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Over (Half, R, J)-separated $\mathcal{A}$ we can express $|R^{\mathcal{A}}| = |J^{\mathcal{A}}|$.

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Over (Half, R, J)-separated $\mathcal{A}$ we can express that $F : R^{\mathcal{A}} \rightarrow J^{\mathcal{A}}$ is functional.

\[
\models \forall x (\exists y F(x, y)) \rightarrow (\exists^{=50\%} y (\text{Half}(y) \land x \neq y) \lor F(x, y))
\]
An instance of SHTP is a system of equations $\varepsilon$ of the form:

\begin{align*}
u &= 1, \\
v &= w + w, \\
vw &= w.
\end{align*}

In SHTP we ask if there is a solution of $\varepsilon$ over $\mathbb{N}$.

For a given $\varepsilon \in \text{SHTP}$ find $\phi_{\varepsilon} \in \text{FO}_2^{\%}$ such that $\phi_{\varepsilon}$ is Finsat iff $\varepsilon$ is solvable.

A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Reduction Part I

1. We use unary predicates $A_u$ for each variable $u$ from $\varepsilon$.

   • Idea: $A_u \models \phi_{\varepsilon}$ then $u \mapsto \text{true}$ is a solution of $\varepsilon$.

2. To encode $u = 1$ we write:

   \begin{align*}
   \exists x A_u(x) \land \forall x \forall y (A_u(x) \land A_u(y) \rightarrow x = y).
   \end{align*}

3. To encode $u = v + w$ we write:

   • For a fresh $\text{Half}$ we write $\exists x\text{Half}(x)$, and that $A_u \subseteq \text{Half}$ as well as $A_v \cup A_w \subseteq \mathbb{N} \setminus \text{Half}$.

   • Employ the trick with equicardinality of $A_u$ and $A_v \cup A_w$.

4. How to encode multiplication?
Definition (Simplified Hilbert’s 10th Problem (SHTP))

An instance of SHTP is a system of equations $\varepsilon$ of the form:

- $u = 1$,
- $u = v + w$,
- $u = v \cdot w$.

In SHTP we ask if there is a solution of $\varepsilon$ over $\mathbb{N}$.
A glance at the undecidability proof of $\mathsf{FO}^2 + \text{global } \%$: Reduction Part I

**Definition (Simplified Hilbert’s 10th Problem (SHTP))**

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For a given $\varepsilon \in \text{SHTP}$ find $\varphi_\varepsilon \in \mathsf{FO}_\%^2$ such that $\varphi_\varepsilon$ is $\text{FinSAT}$ iff $\varepsilon$ is solvable.
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Reduction Part I

**Definition (Simplified Hilbert’s 10th Problem (SHTP))**

An instance of SHTP is a system of equations $\varepsilon$ of the form:

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For a given $\varepsilon \in \text{SHTP}$ find $\varphi_{\varepsilon} \in \text{FO}^2_{\%}$ such that $\varphi_{\varepsilon}$ is FinSAT iff $\varepsilon$ is solvable.

1. We use unary predicates $A_u$ for each variable $u$ from $\varepsilon$. 
A glance at the undecidability proof of $\text{FO}^2 + \text{global }\%$: Reduction Part I

**Definition (Simplified Hilbert’s 10th Problem (SHTP))**

An instance of SHTP is a system of equations $\varepsilon$ of the form:

- $u = 1$,
- $u = v + w$,
- $u = v \cdot w$.

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3. To encode $u = v + w$ we write:
   - For a fresh Half we write $\exists^{=50\%} x. \text{Half}(x)$, and that $A_u^\mathcal{A} \subseteq \text{Half}^\mathcal{A}$ as well as $A_v^\mathcal{A} \cup A_w^\mathcal{A} \subseteq A \setminus \text{Half}^\mathcal{A}$
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**For a given $\varepsilon \in \text{SHTP}$ find $\varphi_\varepsilon \in \text{FO}_{\%}^2$ such that $\varphi_\varepsilon$ is FinSAT iff $\varepsilon$ is solvable.**

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   - For a fresh Half we write $\exists = 50\% x. \text{Half}(x)$, and that $A_u^{\mathcal{A}} \subseteq \text{Half}^{\mathcal{A}}$ as well as $A_v^{\mathcal{A}} \cup A_w^{\mathcal{A}} \subseteq A \setminus \text{Half}^{\mathcal{A}}$
   - Employ the trick with equicardinality of $A_u^{\mathcal{A}}$ and $A_v^{\mathcal{A}} \cup A_w^{\mathcal{A}}$.
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   - Employ the trick with equicardinality of $A_u^{\mathcal{A}}$ and $A_v^{\mathcal{A}} \cup A_w^{\mathcal{A}}$.

4. How to encode multiplication?
A glance at the undecidability proof of FO² + global %: Reduction Part II
A glance at the undecidability proof of \( FO^2 + \text{global \%} \): Reduction Part II

1. To encode \( u \cdot v = w \) (so \( |A_u^{\alpha_1}| \cdot |A_v^{\alpha_1}| = |A_w^{\alpha_1}| \)) we write:
A glance at the undecidability proof of $FO^2 + \text{global } \%$: Reduction Part II

1. To encode $u \cdot v = w$ (so $|A^\vartriangle_u| \cdot |A^\vartriangle_v| = |A^\vartriangle_w|$) we write:
   - Introduce a fresh binary symbol $Mult$. 

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A glance at the undecidability proof of $\mathbf{FO}^2 + \text{global } \%$: Reduction Part II

1. To encode $u \cdot v = w$ (so $|A_u^{2\|} \cdot A_{v^{2\|}} = |A_{w^{2\|}}|$) we write:

- Introduce a fresh binary symbol $\text{Mult}$.
- $\text{Mult}^{2\|}$ links every element from $A_u^{2\|}$ to some elements from $A_{w^{2\|}}$. (easy)
A glance at the undecidability proof of $\text{FO}^2 + \text{global }\%$: Reduction Part II

1. To encode $u \cdot v = w$ (so $|A_u^{\#}| \cdot |A_v^{\#}| = |A_w^{\#}|$) we write:

- Introduce a fresh binary symbol $\text{Mult}$.
- $\text{Mult}^{\#}$ links every element from $A_u^{\#}$ to some elements from $A_w^{\#}$. (easy)
- Every element from $A_u^{\#}$ has exactly $|A_v^{\#}| \cdot \text{Mult}^{\#}$-successors (trick with equi-cardinality).
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Reduction Part II

1. To encode $u \cdot v = w$ (so $|A_u^{2|} \cdot |A_v^{2|} = |A_w^{2|})$ we write:

- Introduce a fresh binary symbol $\text{Mult}$.
- $\text{Mult}^{2|}$ links every element from $A_u^{2|}$ to some elements from $A_w^{2|}$. (easy)
- Every element from $A_u^{2|}$ has exactly $|A_v^{2|}$ $\text{Mult}^{2|}$-successors (trick with equi-cardinality).
- The inverse of $\text{Mult}^{2|}$ is functional (trick with functionality).
A glance at the undecidability proof of $\text{FO}^2 + \text{global } \%$: Reduction Part II

1. To encode $u \cdot v = w$ (so $|A^q_u| \cdot |A^q_v| = |A^q_w|$) we write:

- Introduce a fresh binary symbol $\text{Mult}$.
- $\text{Mult}^q$ links every element from $A^q_u$ to some elements from $A^q_w$. (easy)
- Every element from $A^q_u$ has exactly $|A^q_v|$ $\text{Mult}^q$-successors (trick with equi-cardinality).
- The inverse of $\text{Mult}^q$ is functional (trick with functionality).

By expressing every equation we obtain the desired $\varphi_\varepsilon$ and conclude undecidability.
Summary

• \textsc{Finsat} for \textsc{Fos}\textsuperscript{2} is undecidable under any semantics.

• \textsc{Finsat} for \textsc{Gf}\textsuperscript{2} is undecidable under local semantics.

• \textsc{Finsat} for \textsc{Gf}\textsuperscript{2} + \textsc{global} \% is undecidable.

• \textsc{Finbranchsat} for \textsc{Gf}\textsuperscript{2} + \textsc{local} \% is \textsc{ExpTime}-complete and \textsc{CQ} querying is \textsc{2ExpTime}-complete.

• \textsc{Finsat} for \textsc{Gf}\textsuperscript{2} + \textsc{local} \% is in \textsc{3NExpTime} + \textsc{CQ} querying in \textsc{4NExpTime}.

Open problems

1. How to show \textsc{ExpTime} upper bound for \textsc{finsat} of \textsc{Gf}\textsuperscript{2} + \textsc{local} \%?

2. Can we improve our undecidability proofs to avoid the use of equality symbol?

3. Is there any other decidable logic that will stay decidable with arithmetics?

Any questions?
Summary

- FINSAT for FO^2 + % is undecidable under any semantics.

Open problems

1. How to show ExpTime upper bound for FINSAT of GF^2 + local %?
2. Can we improve our undecidability proofs to avoid the use of equality symbol?
3. Is there any other decidable logic that will stay decidable with arithmetics?
Summary

- FINSAT for $\text{FO}^2 + \%$ is undecidable under any semantics.
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Summary

- FINSAT for $\text{FO}^2 + \%$ is undecidable under any semantics.
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- FINSAT for $\text{GF}^2 + \text{global } \%$ is undecidable.

Open problems
1. How to show $\text{ExpTime}$ upper bound for FINSAT of $\text{GF}^2 + \text{local } \%$?
2. Can we improve our undecidability proofs to avoid the use of equality symbol?
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Summary

• FINSAT for FO\(^2\) + % is **undecidable** under any semantics.
• FINSAT for GF\(^3\) + % is **undecidable** under local semantics.
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Open problems

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Open problems

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