Satisfiability Checking and Conjunctive Query Answering in Description Logics with Global and Local Cardinality Constraints

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Description Logic Workshop 2019
Oslo, June 18th, 2019
Agenda

- Introduction to the logic $ALCSCC^{++}$
- Expressivity examples
- Satisfiability is $\text{NExpTime}$-complete ... 
- but conjunctive query entailment is undecidable
- Nice sub-fragment $ALCSCC$ with decidable finite query answering
QFBAPA

- We recall quantifier-free fragment of Boolean Algebra with Presburger Arithmetic (QFBAPA)
- Set terms, boolean operations $\cap, \cup, \cdot$ on them, constants $\emptyset, \mathcal{U}$
- Set terms can be used to state set constraints, $s = t, s \subseteq t$
- Presburger arithmetic (PA) expressions are build from:
  - Integer constants. $\ldots, -2, -1, 0, 1, 2, \ldots$
  - set cardinalities $|s|$, for $s$ being a set terms
- We can express Cardinality constraints $k = l, k < l, N \text{ dvd } l$

QBFBAPA formula = boolean comb. of set and cardinality constr.
Sat of QFBAPA is NP-complete [Kuncak&Rinard CADE 2007]
The definition of \( \mathcal{ALCSCC}^{++} \)

\( \mathcal{ALCHQ} + \) concepts of the form \( \text{sat}(c) \) for a QFBAPA set/cardinality constraint \( c \) and \( c \) uses role names and \( \mathcal{ALCSCC}^{++} \) concept descriptions in place of set variables

- \( (\geq \text{nr.} C)^I = \text{sat}(|C \cap r| \geq n)^I \) - number restrictions
- \( \text{sat}(|r| \text{ dvd } 2) \) - even number of \( r \) successors
- \( \text{sat}(|\top| \text{ dvd } 2) \) - the total number of elements is even
- \( \text{sat}(|A| = 1) \) - nominals
- \( \text{sat}(\top \subseteq \text{sat}(r \cap s \subseteq \emptyset)) \) - role disjointness
- \( \text{sat}(\top \subseteq \text{sat}(|r| + |r^c| = |U|)) \) - role complementation
- \( \text{sat}(\top \subseteq \text{sat}(|r| = |U|)) \) - universal role
ALCSCC\(^++\) is NExpTime-complete

- We provide an exponential reduction to QFBAPA
- Matching lower bound from previous work [Baader&Ecke, GCAI’17]
- We define a notion of types and write a formula describing them
- \( M = \text{set of all subconcepts from the input concept } E \)
- If \( M \) is a set of concepts, then \( t \subseteq M \) is a type if:
  - If \( \neg C \in M \) then \( C \in t \lor \neg C \in t \)
  - If \( C \sqcap D \in M \) then \( C \sqcap D \in t \iff C \in t \text{ and } D \in t \)
  - If \( C \sqcup D \in M \) then \( C \sqcup D \in t \iff C \in t \text{ or } D \in t \)
- Such a type \( t \) can also be seen as a concept description \( C_t \), which is the conjunction of all the elements of \( t \).
Encoding types in QFBAPA

- Given a type $t$, we replace concepts $C$ with $X_C$ and roles $r$ with $X^t_r$. The resulting formula is $\psi_t$. 

Overall, we translate the concept $E$ into the QFBAPA $\delta_E$:

$$
\delta_E : = (X_C \geq 1) \land \beta \land \left( X_C = 0 \lor \psi_t \right).
$$

First conjunct = witness for $E$

Last two conjuncts = for any type that is realized (i.e., has elements), the constraints of this type are satisfied
Encoding types in QFBAPA

- Given a type \( t \), we replace concepts \( C \) with \( X_C \) and roles \( r \) with \( X_r^t \). The resulting formula is \( \psi_t \).
- We can ensure boolean structure of types with \( \beta := \)

\[
\bigwedge_{C \cap D \in M} X_{C \cap D} = X_C \cap X_D \land \bigwedge_{C \cup D \in M} X_{C \cup D} = X_C \cup X_D \land \bigwedge_{-C \in M} X_{-C} = (X_C)^c
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- Overall, we translate the concept $E$ into the QFBAPA $\delta_E$:

$$\delta_E := (|X_E| \geq 1) \land \beta \land \bigwedge_{t \in \text{types}(E)} (|\bigcap_{C \in t} X_C| = 0) \lor \psi_t.$$
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Conclusion: satisfiability for $\text{ALCSCC}^{++}$

Lemma
The QFBAPA formula $\delta_E$ is of size at most exponential in the size of $E$, and it is satisfiable iff the $\text{ALCSCC}^{++}$ concept description $E$ is satisfiable.
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**Satisfiability of $\mathcal{ALCSCC}^{++}$**
The sat problem for $\mathcal{ALCSCC}^{++}$ is NExpTime-complete.
Query answering is **undecidable**

- **Source of undecidability:** Universal role is expressible in $\mathcal{ALCSCC}^{++}$ (as we have seen before)
- Proof by [Pratt-Hartmann, Inf. Comput. 207] for $\mathcal{FO}^2$ without equality - sketchy, it is not clear whether it is correct
- So we prove it on our own! for $\mathcal{ALC}^{cov}$, i.e., $\mathcal{ALC}$ extended by role cover axioms of the form $\text{cov}(r,s)$
- An interpretation $\mathcal{I}$ satisfies $\text{cov}(r,s)$ if $r^\mathcal{I} \cup s^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.
- Role cover axioms can be expressed in $\mathcal{ALCSCC}^{++}$ via
  \[
  \text{sat}(\top \subseteq \text{sat}(|r \cup s| = |\mathcal{U}|))
  \]
- Reduction from the looping Turing machines, i.e. it is undecidable whether DTM is looping
How decidability could be regain?

- We move to a less expressive, e.g., $\textit{ALCSCC}$ with RCBoxes.
- In $\textit{ALCSCC}$ all QFBAAPA constraints must be local!
  It is equivalent to write:
  \[
  C_{\text{new}} = C_{\text{old}} \cap ( \bigcup_{r \in N_R} r )
  \]
- RCBoxes = finite sets of restricted cardinality constraints
  \[
  N_1|C_1| + \ldots + N_k|C_k| \leq N_{k+1}|C_{k+1}| + \ldots + N_{k+l}|C_{k+l}|,
  \]
- Not able to express nominals!
  But still useful to express statistical knowledge-bases.
Decidable query answering

- We reduce query entailment to satisfiability
- We enrich our knowledge-base with ability to block all tree-shaped query matches (so-called rolling-up technique)
- Then we employ pumping technique to obtain models with arbitrary girth, while preserving satisfiability
- So if there is a counter-model, there is be a model without query tree-shaped matches
Blocking tree-shaped query matches

- We take an arbitrary CQ $q$.
- Consider all the possible ways $q'$ how $q$ can match as a tree.
- We create a concept $C_{q',x}$, with the supposed meaning that $d \in C_{q',x}$ if variable $x$ from $q'$ can be mapped to $d$ in a query match represented by $q'$.
- We roll them into concepts in bottom-up way:
- $C_{q',x}$ equals $\bigcap_{C(x) \in q'} C$ if $x$ is a leaf (i.e. $\prec$-minimal), otherwise:

$$\bigcap_{C(x) \in q'} C \cap \bigcap_{(x,y) \in E_{q'}} \left( \exists \bigcap_{s(x,y) \in q'} s . C_{q',y} \right) \cap \bigcap_{(y,x) \in E_{q'}} \left( \exists \bigcap_{s(y,x) \in q'} s^{-} . C_{q',y} \right)$$
Correctness

We define $R_{\text{Match}_q}$ as:

$$
\bigcup_{q' \in \text{trees}(q)} C_{q',x^r_{q'}} \subseteq \text{Match}_q
$$

Lemma
Assume that $R \cup R_{q_{\text{Match}}}$ has a model $\mathcal{I}$ such that $\text{Match}_q^\mathcal{I}$ is empty. Then $\mathcal{I}$ does not have any tree-shaped query matches.

Lemma
If there is a model $\mathcal{I}$ of $R$ without any tree-shaped query matches, then $R^* = R \cup R_{q_{\text{Match}}} \cup \{ \top \subseteq \neg \text{Match}_q \}$ is satisfiable.
Pumping lemma for graphs

The girth of $\mathcal{I}$ is the length of a shortest (undirected) proper cycle contained in $\Delta^\mathcal{I}$. Below we present pumping method for graphs:
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Let $G = (V, E)$ be a graph, $F$ be the set of functions $f : E \to \{0, 1\}$. Construct $H = (V', E')$ as follows:

- $V' = V \times F$
- $E'$ be the set of edges $((u, f), (u', f'))$ such that $e = (u, u')$ is in edge of $E$ and $f'$ is the same as $f$ except that the value at $e$ is flipped: $f'(e) = 1 - f(e)$.

Then, the girth of $H$ is at twice that of $G$.

Easy to generalize to structures.
Properties of pumping

Lemma
Let $\mathcal{I}$ be an interpretation with girth $k$.
Then the girth of $\text{pump}(\mathcal{I})$ is at least $2k$.

Lemma
Let $\mathcal{R}$ be an RCBox with a model $\mathcal{I}$.
Then $\text{pump}(\mathcal{I})$ is a model of $\mathcal{R}$. 
Decidable query answering

- We check enriched kb for satisfiability.
- If it is satisfiable, it does not have any tree-shaped query matches.
- We pump it at least $|q|$ times to obtain a countermodel.
- Thus satisfiable $= \text{query is not entailed.}$

**Querying for $\mathcal{ALCSCC}$ RCBoxes**

Conjunctive Querying is decidable for $\mathcal{ALCSCC}$ RCBoxes.
Conclusion

**Satisfiability and querying of ALCSCC++**

The sat problem for ALCSCC++ is NExpTime-complete, but query-answering is undecidable.

**Querying for ALCSCC RCBoxes**

Conjunctive Querying is decidable for ALCSCC RCBoxes.

We also know how to add ABoxes. We are working on improving the complexity (seems to be doable) of CQ entailment.

Open problems? The case with nominals!