On Classical Decidable Logics Extended with Percentage Quantifiers and Arithmetics

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Motivating example: election knowledge-bases

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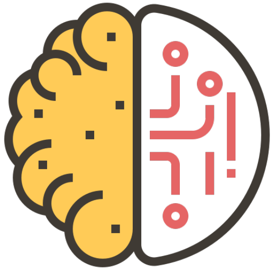
Database

Knowledge



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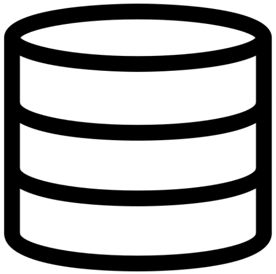

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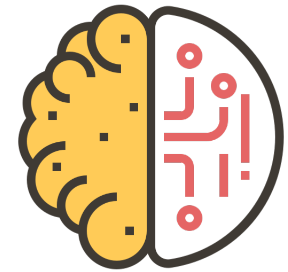

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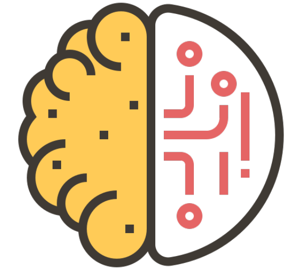

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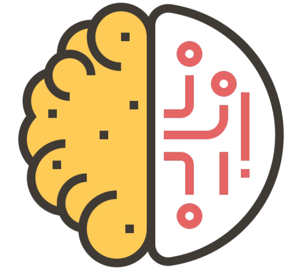

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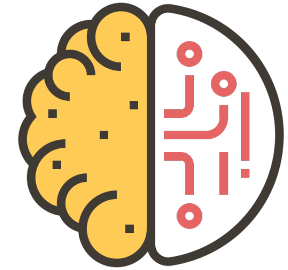

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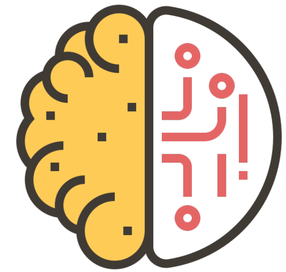

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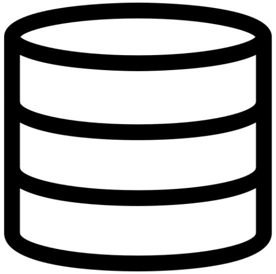

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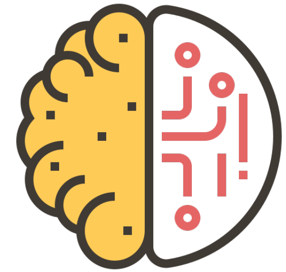

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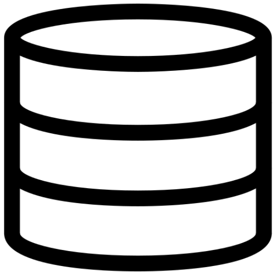

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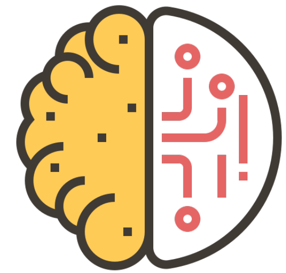

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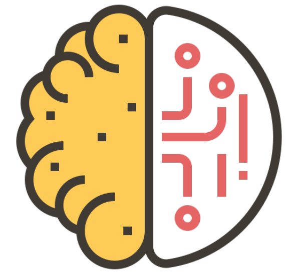

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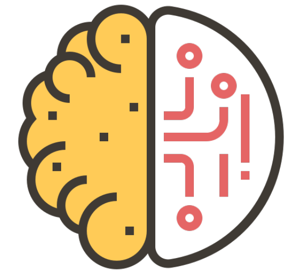

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Related work

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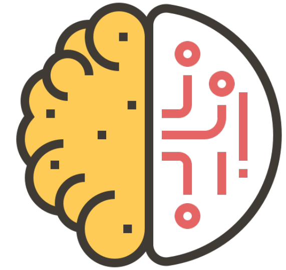

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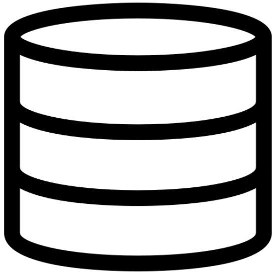

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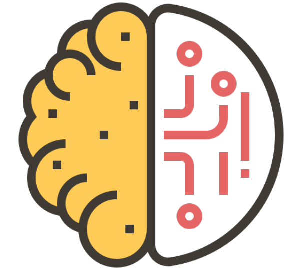

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4. $\mathcal{ALCISCC}^{++}$ [Baader et. al'2020] or FO^2 with Härtig quantifier [Grädel et al.'1999]

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Example: Some artist admires only beekeepers

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Coexample: Every artist admires every beekeeper

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So let's add them! Why not?

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- FINSAT for $\mathsf{FO}^2 + \%$ is undecidable under any semantics.
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Our positive results hold even for Presburger's arithmetic (FO[+]) constraints on successors.

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- GF^3 + functional role is undecidable [Grädel&Otto&Rosen'1999].
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3. Decidability of CQ query entailment

- Exponential reduction to satisfiability, based on "pumping" from [Baader&B.&Rudolph, DL'2019].

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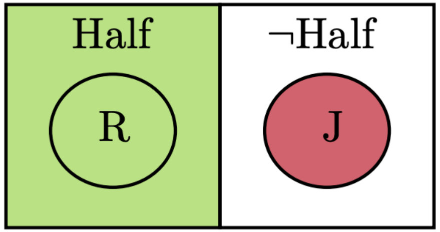

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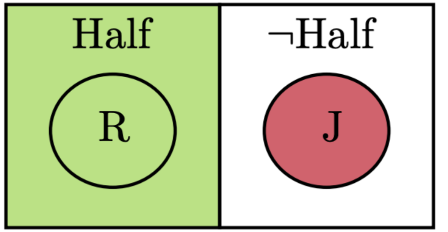

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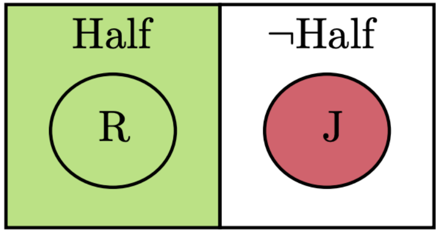

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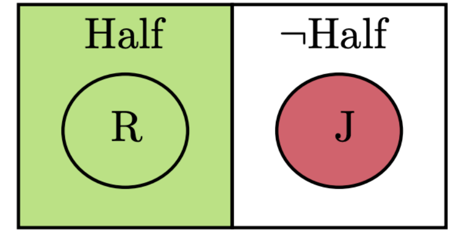

Over $(\mathrm{Half}, \mathrm{R}, \mathrm{J})$-separated $\mathfrak{A}$ we can express $|\mathrm{R}^{\mathfrak{A}}| = |\mathrm{J}^{\mathfrak{A}}|$.

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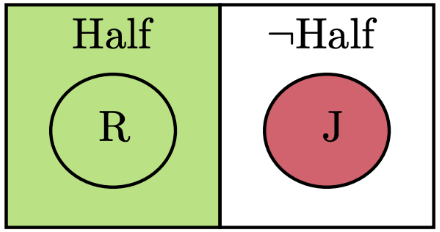

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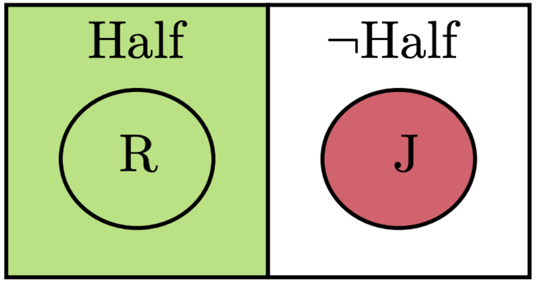

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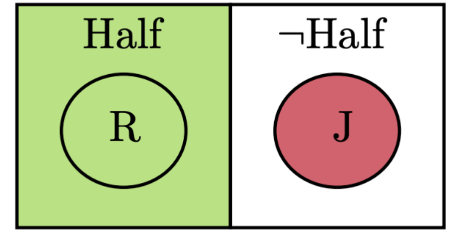

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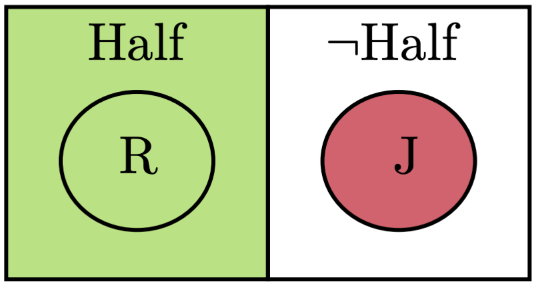

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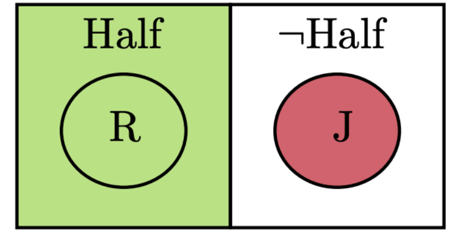

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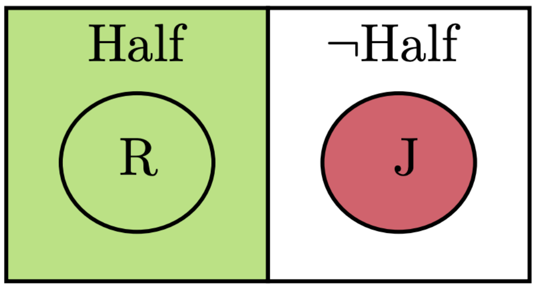

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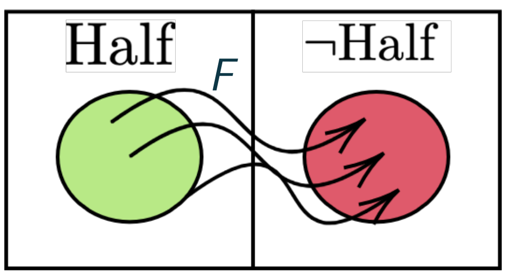

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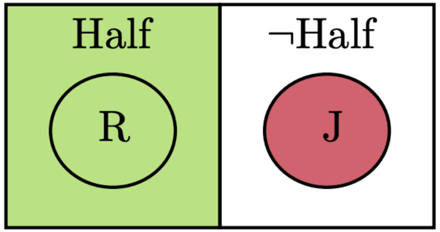

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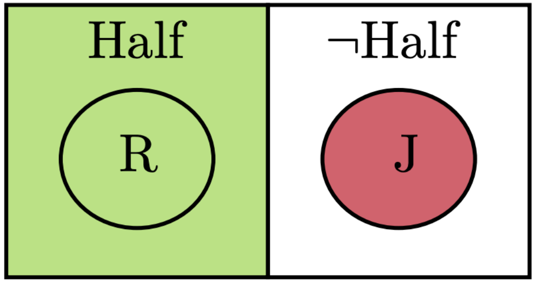

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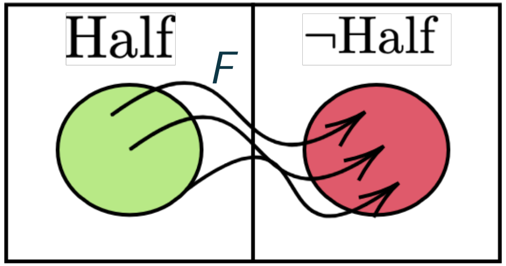

$$\models \forall x(\exists y F(x,y)) \to (\exists^{=50\%} y\ (\mathrm{Half}(y) \land x \neq y) \lor F(x,y))$$

A glance at the undecidability proof of FO^2 + global %: Reduction Part I

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4. How to encode multiplication?

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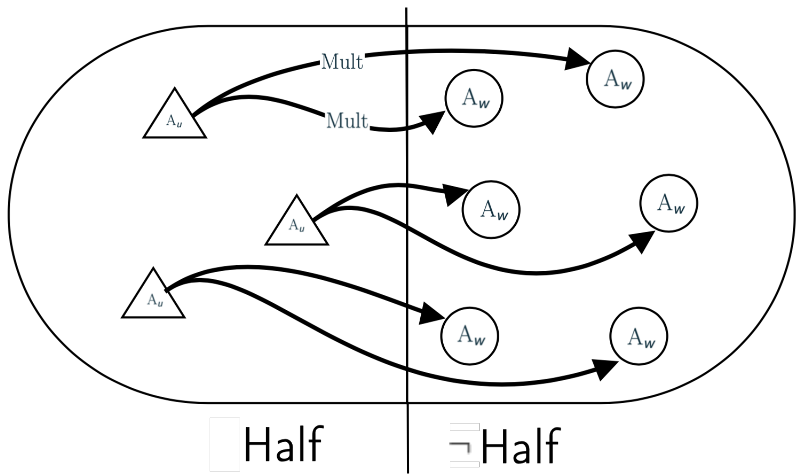

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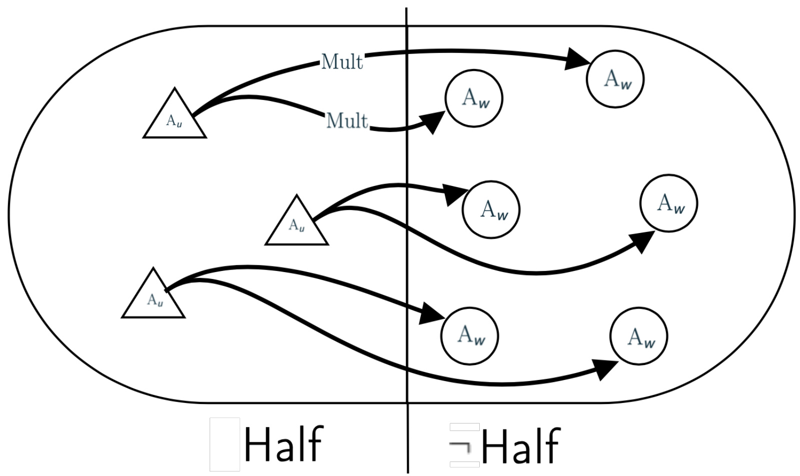

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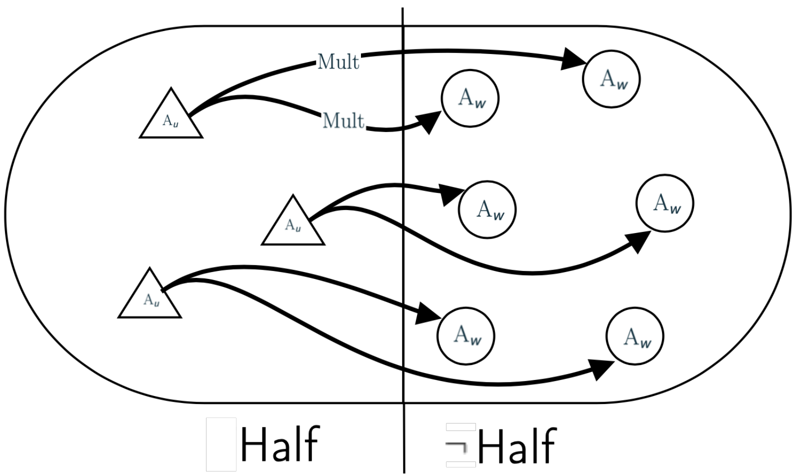

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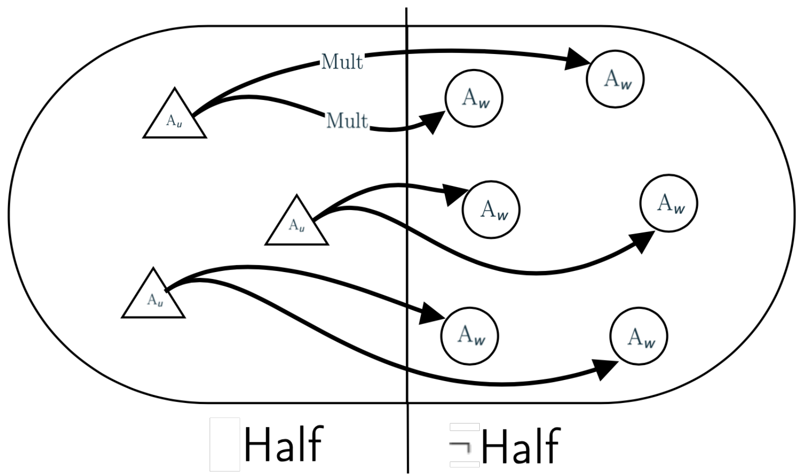

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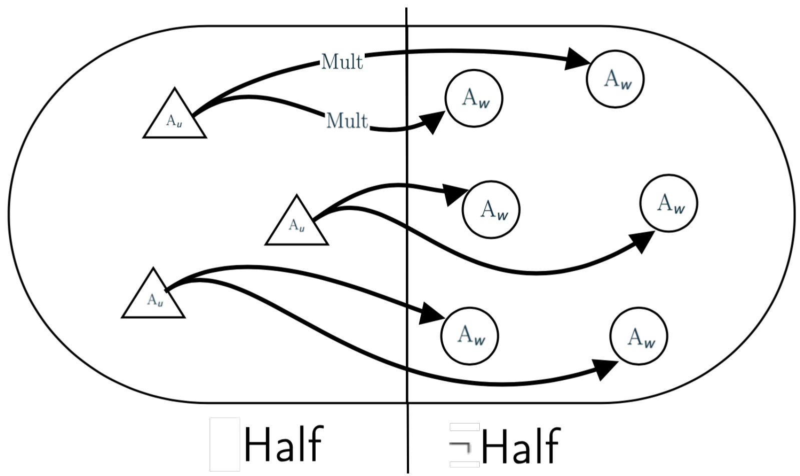

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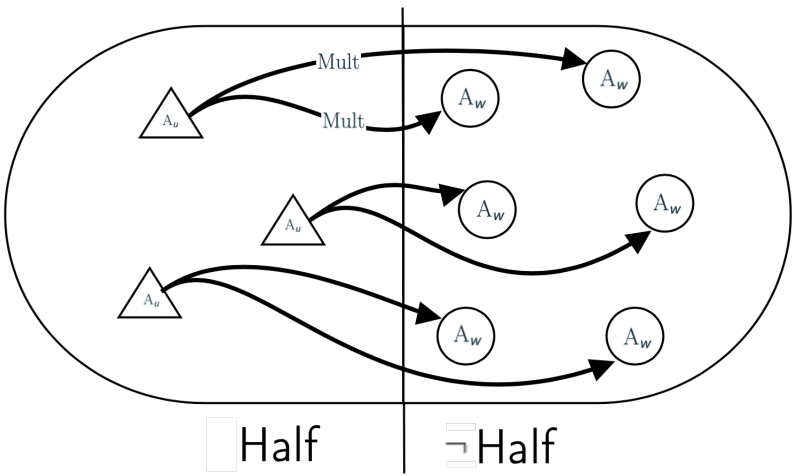

By expressing every equation we obtain the desired φ_ε and conclude undecidability.

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