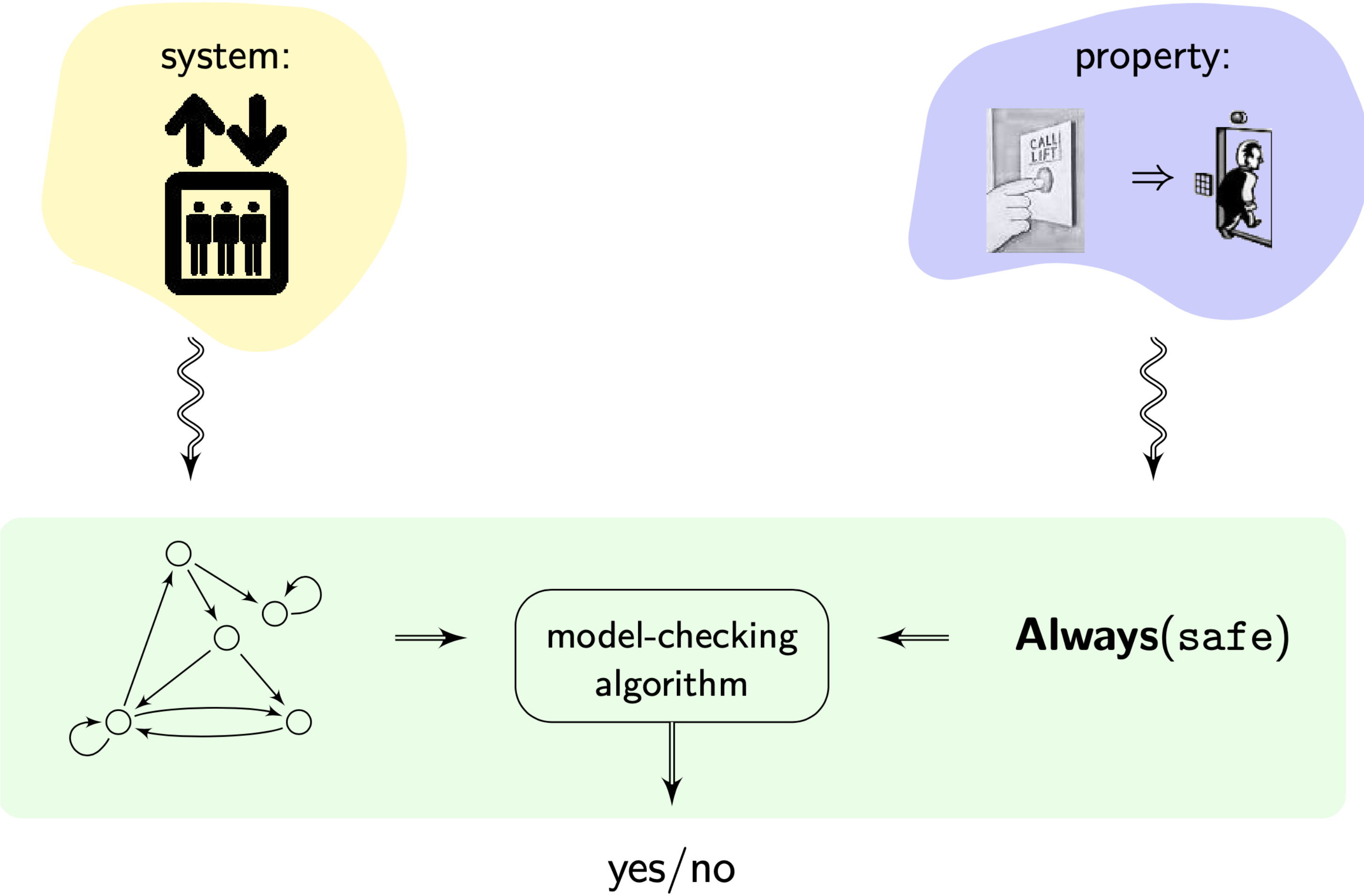
"Most of" leads to undecidability
Failure of adding frequencies to LTL

FoSSaCS 2021

Bartosz Bednarczyk, Jakub Michaliszyn

TU Dresden & University of Wrocław

What's the formal verification about?

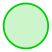
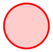


Linear-time Temporal Logic (LTL)

Linear-time Temporal Logic (LTL)

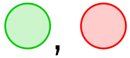
- **atomic propositions:** , , …

Linear-time Temporal Logic (LTL)

- **atomic propositions:** , , ...

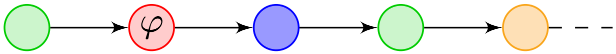
- **boolean combinators:** $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, ...

Linear-time Temporal Logic (LTL)

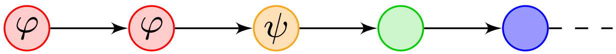
- **atomic propositions:** , …

- **boolean combinators:** $\neg\,\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, …

- **temporal modalities:**

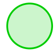
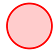
$\mathbf{X}\,\varphi$ 

"next φ"

$\varphi\ \mathbf{U}\ \psi$ 

"φ until ψ"

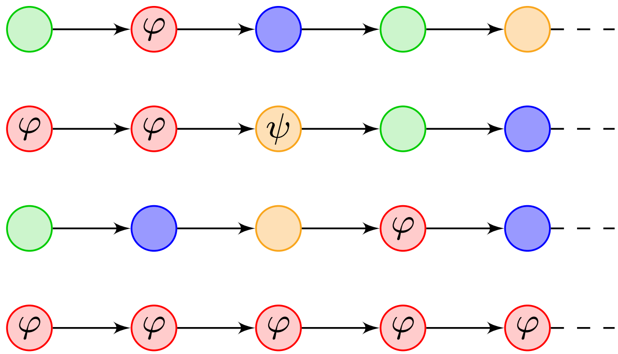
Linear-time Temporal Logic (LTL)

- **atomic propositions:** , , ...

- **boolean combinators:** $\neg\,\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, ...

- **temporal modalities:**

$\mathbf{X}\,\varphi$	“next φ”
$\varphi\ \mathbf{U}\ \psi$	“φ until ψ”
$\mathtt{true}\ \mathbf{U}\ \varphi \equiv \mathbf{F}\,\varphi$	“eventually φ”
$\neg\ \mathbf{F}\ \neg\,\varphi \equiv \mathbf{G}\,\varphi$	“always φ”



Satisfiability and model checking

Two main algorithmic problems

Satisfiability and model checking

Two main algorithmic problems

- **Satisfiability:**
  - Input: a formula φ in LTL;
  - Output:

    yes if there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \varphi$;

    no otherwise.

Satisfiability and model checking

Two main algorithmic problems

- **Satisfiability:**
  - Input: a formula φ in LTL;
  - Output:

    yes if there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \varphi$;

    no otherwise.

- **Model checking:**
  - Input: a formula φ in LTL, and a Kripke structure $\mathcal{M}$;
  - Output:

    yes if $\mathcal{M} \models \varphi$;

    no otherwise.

LTL: Ups and downs

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

- Model checking and satisfiability are logspace interreducible.

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

- Model checking and satisfiability are logspace interreducible.
- PSpace upper bound = on-the-fly construction of Buchi automata

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

- Model checking and satisfiability are logspace interreducible.
- PSpace upper bound $=$ on-the-fly construction of Buchi automata

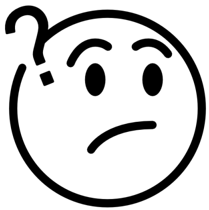
LTL is useful in verification and has good algorithmic properties

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

- Model checking and satisfiability are logspace interreducible.
- PSpace upper bound = on-the-fly construction of Buchi automata

LTL is useful in verification and has good algorithmic properties



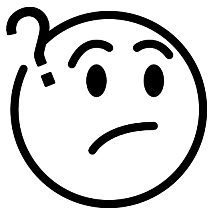
So what's wrong with it?

LTL: Ups and downs

Theorem (LTL is PSPACE-complete.)

- Model checking and satisfiability are logspace interreducible.
- PSpace upper bound = on-the-fly construction of Buchi automata

LTL is useful in verification and has good algorithmic properties



So what's wrong with it?

but it can't express quantitative properties!

Our goal: Extend LTL with frequency constraints

Our goal: Extend LTL with frequency constraints

Related works:

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] $=$ long paths makes formulae less true.

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] = long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] $=$ long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] $=$ long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].
- Availability expressions [Hoenicke et al. 2010]:

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] = long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].
- Availability expressions [Hoenicke et al. 2010]:

"Kleene, Rabin, and Scott are available"

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] = long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].
- Availability expressions [Hoenicke et al. 2010]:

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] $=$ long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].
- Availability expressions [Hoenicke et al. 2010]:

All of them are undecidable!

Our goal: Extend LTL with frequency constraints

Related works:

- Frequency LTL [Bollig et al. 12]: Extension of LTL with frequency until.
- Averaging in LTL [Bouyer et al. 14]: weighted alphabet, until calculating avg.
- Discounted-LTL [Almagor et al. 14] = long paths makes formulae less true.
- Metric LTL [Koymans'90] - time modelled as a real line
- ULTL[F,P, X, Y] with Presburger Arithmetics [Lodaya and Sreejith 17].
- Availability expressions [Hoenicke et al. 2010]:

All of them are undecidable!

And the problem seems to be the until operator.

Our setting

Our setting

We allow only for "finally" **F** operator $+$

Our setting

We allow only for “finally” **F** operator $+$

“most of the previous positions satisfies φ” $\mathbf{PM}(\varphi)$

Our setting

We allow only for "finally" **F** operator +

"most of the previous positions satisfies φ" $\mathbf{PM}(\varphi)$

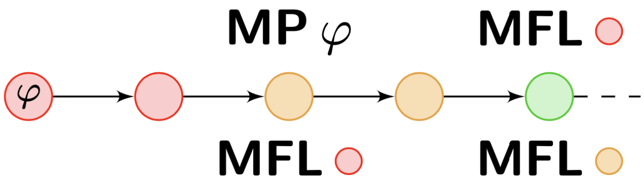
or "a is the most-frequent-letter in the past" $\mathbf{MFL}(a)$

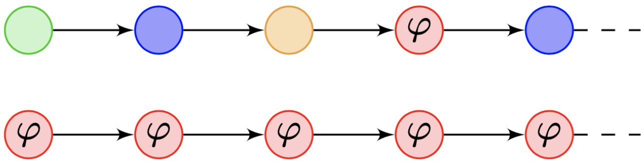
Our setting

We allow only for "finally" **F** operator +

"most of the previous positions satisfies φ" $\mathbf{PM}(\varphi)$

or "a is the most-frequent-letter in the past" $\mathbf{MFL}(a)$





$\mathbf{F}\,\varphi$ "eventually φ"

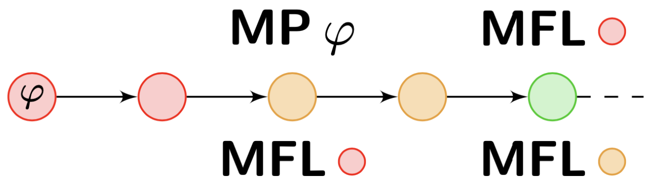
$\neg\,\mathbf{F}\,\neg\varphi \equiv \mathbf{G}\,\varphi$ "always φ"

Our setting

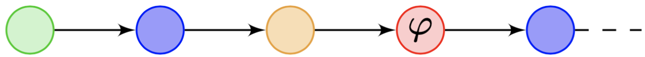
We allow only for "finally" **F** operator +

"most of the previous positions satisfies φ" $\mathbf{PM}(\varphi)$

or "a is the most-frequent-letter in the past" $\mathbf{MFL}(a)$

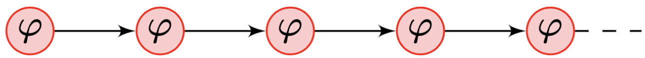


$\mathbf{F}\,\varphi$



"eventually φ"

$\neg\,\mathbf{F}\,\neg\varphi \equiv \mathbf{G}\,\varphi$



"always φ"

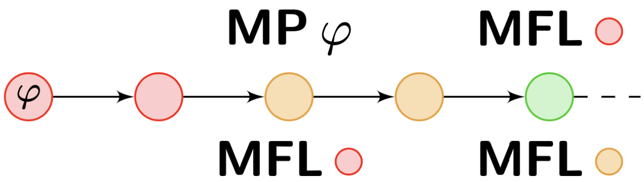
$\mathfrak{w}, i \models \mathbf{F}\,\varphi$ if $\exists j$ such that $|\mathfrak{w}| > j \geq i$ and $\mathfrak{w}, j \models \varphi$

Our setting

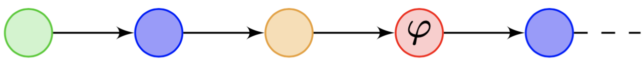
We allow only for "finally" **F** operator +

"most of the previous positions satisfies φ" $\mathbf{PM}(\varphi)$

or "a is the most-frequent-letter in the past" $\mathbf{MFL}(a)$

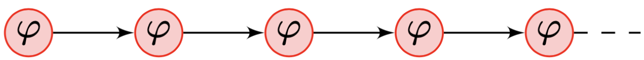


$\mathbf{F}\,\varphi$



"eventually φ"

$\neg\,\mathbf{F}\,\neg\varphi \equiv \mathbf{G}\,\varphi$



"always φ"

$$\mathfrak{w}, i \models \mathbf{F}\,\varphi \quad \text{if} \quad \exists j \text{ such that } |\mathfrak{w}| > j \geq i \text{ and } \mathfrak{w}, j \models \varphi$$

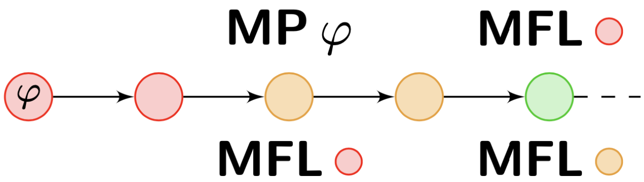
$$\mathfrak{w}, i \models \mathbf{PM}\,\varphi \quad \text{if} \quad |\{j < i \colon \mathfrak{w}, j \models \varphi\}| \geq \tfrac{i}{2}$$

Our setting

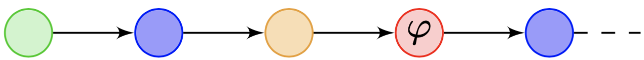
We allow only for "finally" **F** operator +

"most of the previous positions satisfies φ" $\mathbf{PM}(\varphi)$

or "a is the most-frequent-letter in the past" $\mathbf{MFL}(a)$

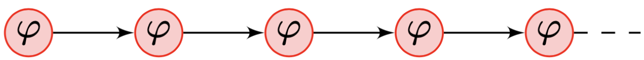


$\mathbf{F}\,\varphi$



"eventually φ"

$\neg\,\mathbf{F}\,\neg\varphi \equiv \mathbf{G}\,\varphi$



"always φ"

$$\mathfrak{w}, i \models \mathbf{F}\,\varphi \quad \text{if} \quad \exists j \text{ such that } |\mathfrak{w}| > j \geq i \text{ and } \mathfrak{w}, j \models \varphi$$

$$\mathfrak{w}, i \models \mathbf{PM}\,\varphi \quad \text{if} \quad |\{j < i : \mathfrak{w}, j \models \varphi\}| \geq \tfrac{i}{2}$$

$$\mathfrak{w}, i \models \mathbf{MFL}\,\sigma \quad \text{if} \quad \forall \tau \in \mathrm{AP}.\ |\{j < i : \mathfrak{w}, j \models \sigma\}| \geq |\{j < i : \mathfrak{w}, j \models \tau\}|$$

Our results

Our results

- LTL with **F** and **PM** is undecidable.

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of $\mathrm{LTL}+$**PM** are decidable.

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of $\mathrm{LTL}+$**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of LTL+**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of LTL+**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of $\mathrm{LTL}+$**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

$$\mathfrak{w}, i \models \mathbf{Half}\,\varphi \text{ if } |\{j < i : \mathfrak{w}, j \models \varphi\}| = \frac{i}{2}$$

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of LTL+**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

$$\mathfrak{w}, i \models \textbf{Half}\,\varphi \text{ if } |\{j < i \colon \mathfrak{w}, j \models \varphi\}| = \frac{i}{2}$$

$$\textbf{Half}\,\varphi := \textbf{PM}\,(\varphi) \wedge \textbf{PM}\,(\neg\varphi)$$

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of $\mathrm{LTL}+$**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

$$\mathfrak{w}, i \models \textbf{Half}\,\varphi \text{ if } |\{j < i \colon \mathfrak{w}, j \models \varphi\}| = \tfrac{i}{2}$$

$$\textbf{Half}\,\varphi := \textbf{PM}\,(\varphi) \wedge \textbf{PM}\,(\neg\varphi)$$

- The proof goes via encoding of Minsky's two counter machines

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of $\mathrm{LTL}+$**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

$$\mathfrak{w}, i \models \textbf{Half}\,\varphi \text{ if } |\{j < i \colon \mathfrak{w}, j \models \varphi\}| = \frac{i}{2}$$

$$\textbf{Half}\,\varphi := \textbf{PM}\,(\varphi) \land \textbf{PM}\,(\neg\varphi)$$

- The proof goes via encoding of Minsky's two counter machines

In the last few minutes we present the main ideas of the encoding.

Shadowy words

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

Shadowy words

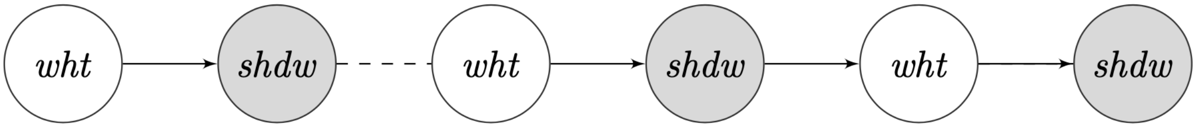
Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^+$

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

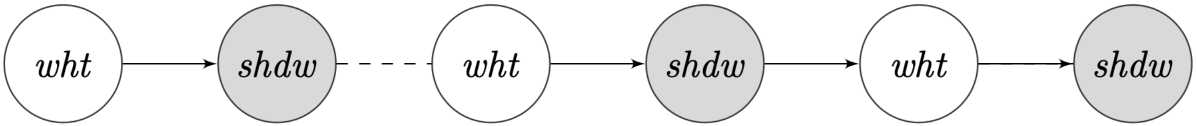
A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^+$



Shadowy words

Consider an alphabet $\{\mathit{wht}, \mathit{shdw}\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(\mathit{wht} \cdot \mathit{shdw})^+$



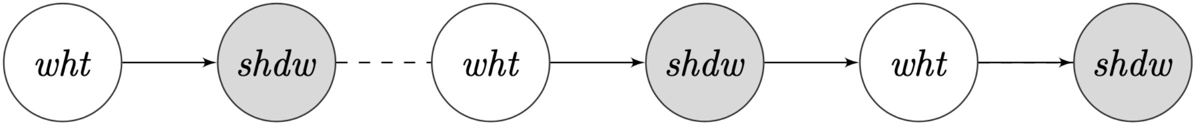
Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^{+}$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

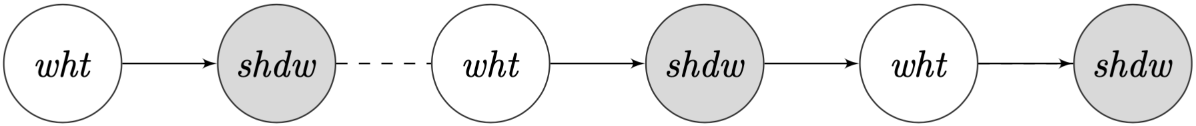
Proof

It suffices to employ the following formulae:

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^{+}$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

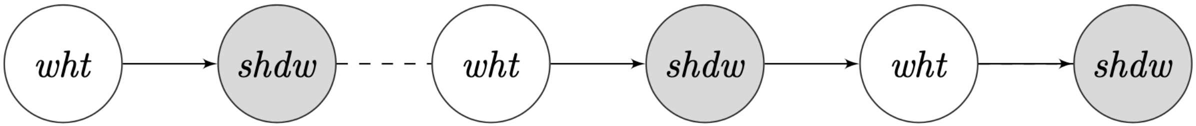
It suffices to employ the following formulae:

- *wht*

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^+$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

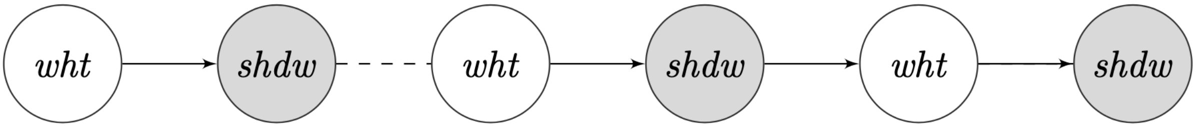
It suffices to employ the following formulae:

- wht
- $\mathbf{G}\,(wht \leftrightarrow \neg shdw)$

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^+$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

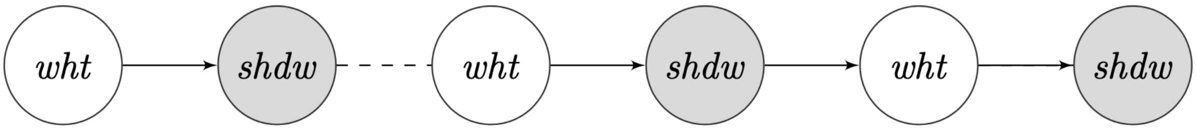
It suffices to employ the following formulae:

- wht
- $\mathbf{G}\,(wht \leftrightarrow \neg shdw)$
- $\mathbf{G}\,(wht \rightarrow \mathbf{F}\,(shdw))$

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^{+}$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

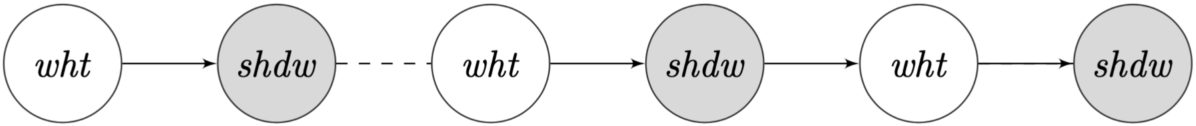
It suffices to employ the following formulae:

- wht
- $\mathbf{G}\left(wht \leftrightarrow \neg shdw\right)$
- $\mathbf{G}\left(wht \rightarrow \mathbf{F}\left(shdw\right)\right)$
- $\mathbf{G}\left(\varphi_{even} \leftrightarrow wht\right)$

Shadowy words

Consider an alphabet $\{wht, shdw\}$.

A word $\mathfrak{w}$ is shadowy it belongs to $(wht \cdot shdw)^{+}$



Lemma

Shadowy words are $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

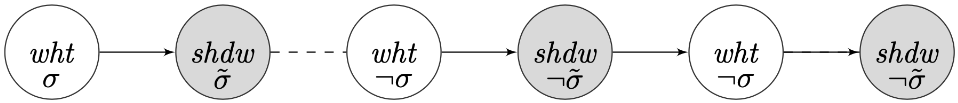
It suffices to employ the following formulae:

- wht
- $\mathbf{G}\,(wht \leftrightarrow \neg shdw)$
- $\mathbf{G}\,(wht \rightarrow \mathbf{F}\,(shdw))$
- $\mathbf{G}\,(\varphi_{even} \leftrightarrow wht)$, where $\varphi_{even} := \mathbf{Half}\; wht$

Transferring truth predicates

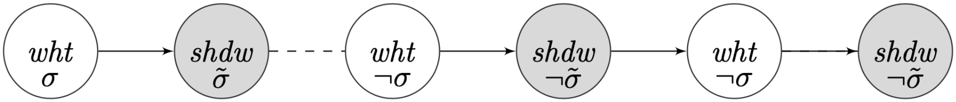
Exercise 3.3. *Let σ and $\tilde{\sigma}$ be distinct letters from* $\mathsf{AP} \setminus \{wht, shdw\}$. *There is an* $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$ *formula* $\varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$, *such that* $\mathfrak{w} \models \varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$ *iff:*

1. $\mathfrak{w}$ *is shadowy,*
2. *only white (resp., shadow) positions of* $\mathfrak{w}$ *can be labelled* σ *(resp.,* $\tilde{\sigma}$*) and*
3. *for any even position* p *we have:* $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.



Exercise 3.3. *Let σ and $\tilde{\sigma}$ be distinct letters from* $\mathsf{AP} \setminus \{wht, shdw\}$. *There is an* $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$ *formula* $\varphi^{trans}_{\sigma\rightsquigarrow\tilde{\sigma}}$, *such that* $\mathfrak{w} \models \varphi^{trans}_{\sigma\rightsquigarrow\tilde{\sigma}}$ *iff:*

1. *$\mathfrak{w}$ is shadowy,*
2. *only white (resp., shadow) positions of $\mathfrak{w}$ can be labelled σ (resp., $\tilde{\sigma}$) and*
3. *for any even position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.*

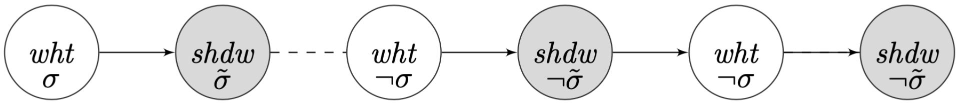


Lemma

Transfer formulae $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Exercise 3.3. *Let σ and $\tilde{\sigma}$ be distinct letters from* AP $\setminus \{wht, shdw\}$. *There is an* $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$ *formula* $\varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$, *such that* $\mathfrak{w} \models \varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$ *iff:*

1. $\mathfrak{w}$ *is shadowy,*
2. *only white (resp., shadow) positions of* $\mathfrak{w}$ *can be labelled* σ *(resp.,* $\tilde{\sigma}$*) and*
3. *for any even position* p *we have:* $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.



Lemma

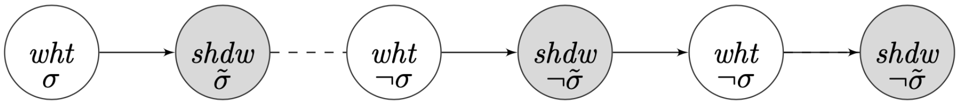
Transfer formulae $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

It suffices to express:

Exercise 3.3. *Let σ and $\tilde{\sigma}$ be distinct letters from* $\mathsf{AP} \setminus \{wht, shdw\}$. *There is an* $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$ *formula* $\varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$, *such that* $\mathfrak{w} \models \varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$ *iff:*

1. $\mathfrak{w}$ *is shadowy,*
2. *only white (resp., shadow) positions of* $\mathfrak{w}$ *can be labelled* σ *(resp.,* $\tilde{\sigma}$*) and*
3. *for any even position p we have:* $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.



Lemma

Transfer formulae $\mathrm{LTL}_{\mathsf{F},\mathsf{Half}}$-definable.

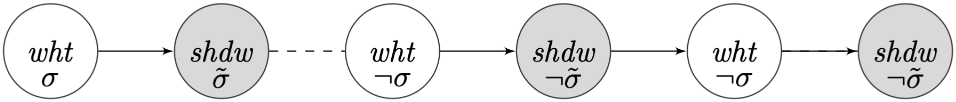
Proof

It suffices to express:

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Exercise 3.3. *Let σ and $\tilde{\sigma}$ be distinct letters from* $\mathsf{AP} \setminus \{wht, shdw\}$*. There is an* $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$ *formula* $\varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$*, such that* $\mathfrak{w} \models \varphi^{trans}_{\sigma \rightsquigarrow \tilde{\sigma}}$ *iff:*

1. *$\mathfrak{w}$ is shadowy,*
2. *only white (resp., shadow) positions of $\mathfrak{w}$ can be labelled σ (resp., $\tilde{\sigma}$) and*
3. *for any even position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.*



Lemma

Transfer formulae $\mathrm{LTL}_{\mathbf{F},\mathbf{Half}}$-definable.

Proof

It suffices to express:

- $(\diamondsuit)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.
- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

- $(\lozenge)$:   for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows!

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

  Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

- $(\diamondsuit)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

  Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

  Second to last position is white:

- $(\diamondsuit)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

Second to last position is white: wht ...

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

Second to last position is white: $wht \ \ldots$

and sees only the last shadows

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

Second to last position is white: $wht\ \ldots$

and sees only the last shadows $\mathbf{G}\,(shdw \rightarrow \varphi_{last})$

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

Second to last position is white: $wht\ \ldots$

and sees only the last shadows $\mathbf{G}\,(shdw \rightarrow \varphi_{last})$

Hence, take $\varphi_{sec\text{-}to\text{-}last} := wht \wedge \mathbf{G}\,(shdw \rightarrow \varphi_{last})$

- $(\Diamond)$: for the last white position p we have: $\mathfrak{w}, p \models \sigma \Leftrightarrow \mathfrak{w}, p{+}1 \models \tilde{\sigma}$.

Last position sees only shadows! $\varphi_{last} := \mathbf{G}\,(shdw)$

Second to last position is white: $wht \ \ldots$

and sees only the last shadows $\mathbf{G}\,(shdw \rightarrow \varphi_{last})$

Hence, take $\varphi_{sec\text{-}to\text{-}last} := wht \wedge \mathbf{G}\,(shdw \rightarrow \varphi_{last})$

and the formula $\mathbf{F}\,(\varphi_{sec\text{-}to\text{-}last} \wedge \pm\sigma) \wedge \mathbf{F}\,(\varphi_{last} \wedge \pm\tilde{\sigma})$ does the job!

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = 0$$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = 0$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \frac{p}{2} - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \frac{p}{2} = \text{"Half"}$$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = 0$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \frac{p}{2} - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \frac{p}{2} = \text{"Half"}$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \#^{<}_{shdw}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \text{"Half"}$$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = 0$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \frac{p}{2} - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \frac{p}{2} = \text{"Half"}$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \#^{<}_{shdw}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \text{"Half"}$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \#^{<}_{shdw \wedge \neg\tilde{\sigma}}(\mathfrak{w}, p) = \text{"Half"}$$

- all white p satisfy $(\heartsuit)$: $\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) = \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p)$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = 0$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \frac{p}{2} - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \frac{p}{2} = \text{"Half"}$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \#^{<}_{shdw}(\mathfrak{w}, p) - \#^{<}_{shdw \wedge \tilde{\sigma}}(\mathfrak{w}, p) = \text{"Half"}$$

$$\#^{<}_{wht \wedge \sigma}(\mathfrak{w}, p) + \#^{<}_{shdw \wedge \neg\tilde{\sigma}}(\mathfrak{w}, p) = \text{"Half"}$$

and hence we get a formula $\textbf{Half}([wht \wedge \sigma] \vee [shdw \wedge \neg\tilde{\sigma}])$

Our results

- LTL with **F** and **PM** is undecidable.
- LTL with **F** and **MFL** is undecidable.
- Some rather uninteresting fragments of LTL+**PM** are decidable.
- $\mathrm{FO}^2[<] +$ Majority quantifier is undecidable.

Our proof technique

- We focus on a single modality **Half**:

$$\mathfrak{w}, i \models \mathbf{Half}\, \varphi \text{ if } |\{j < i \colon \mathfrak{w}, j \models \varphi\}| = \frac{i}{2}$$

$$\mathbf{Half}\, \varphi := \mathbf{PM}\,(\varphi) \wedge \mathbf{PM}\,(\neg\varphi)$$

- The proof goes via encoding of Minsky's two counter machines
- We use shadowy words and tricks with $+\frac{p}{2}$ to express equicardinality

Thanks for attention!

Some initial LTL slides by ©Nicolas Markey.