Satisfiability and Query Answering in Description Logics

with Global and Local Cardinality Constraints

Franz Baader, Bartosz Bednarczyk, Sebastian Rudolph

TU Dresden & University of Wrocław

Running example: Greek mythology $\mathcal{ALCQ}$ knowledge base

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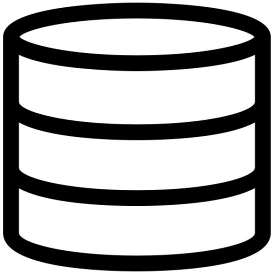
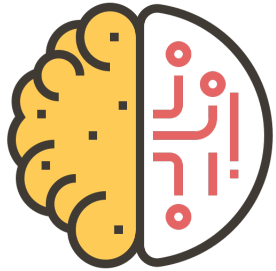
Database (ABox)



Knowledge (TBox)



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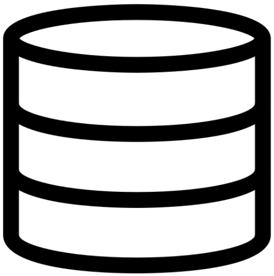

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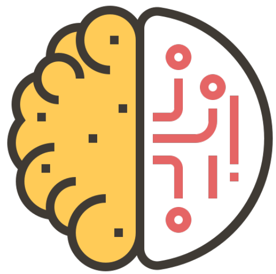

hasParent($\texttt{Heracles}$, $\texttt{Zeus}$)

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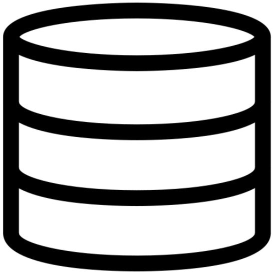

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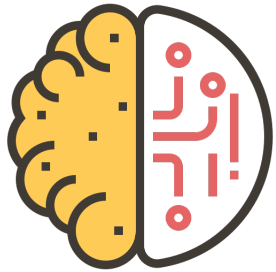

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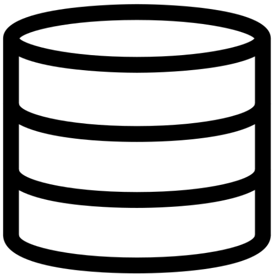

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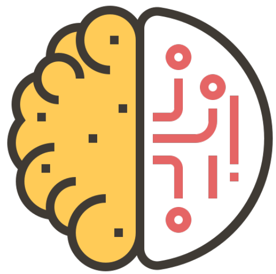

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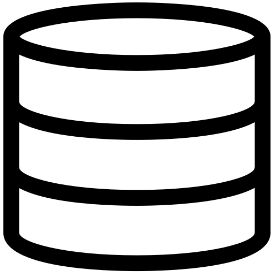

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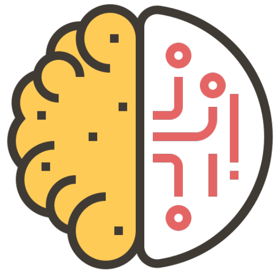

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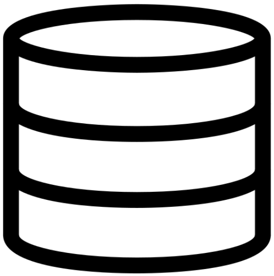

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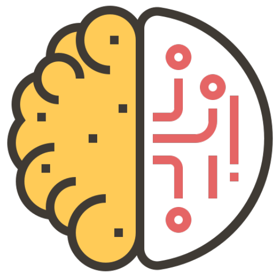

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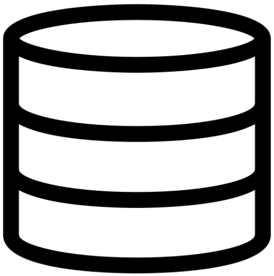

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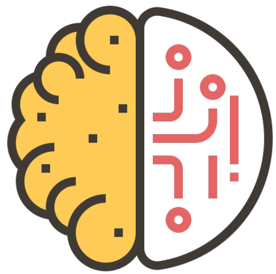

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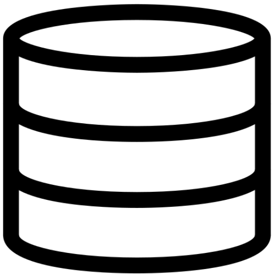

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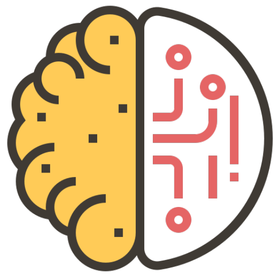

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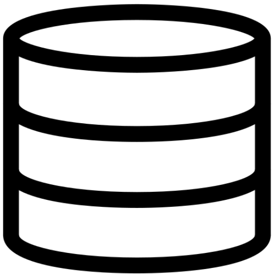

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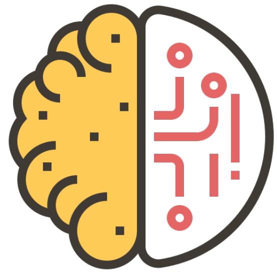
Main problem: Counting

Knowledge (TBox)



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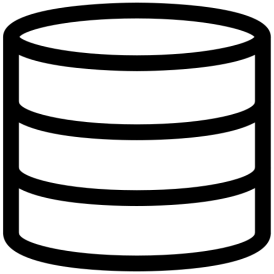

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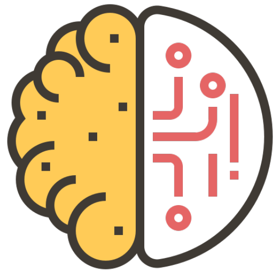

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Main problem: Counting is very-limited and local.

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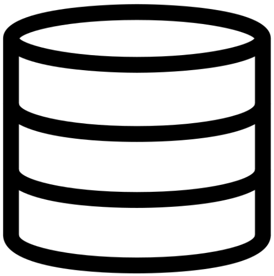

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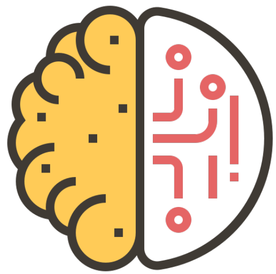

$Mortal \sqsubseteq \neg Diety$

$\top \sqsubseteq \exists hasParent.Male \sqcap \exists hasParent.Female$

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Main problem: Counting is very-limited and local. How to express:

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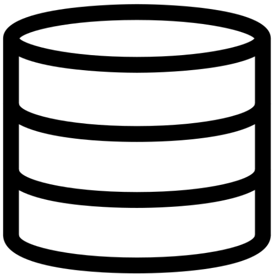

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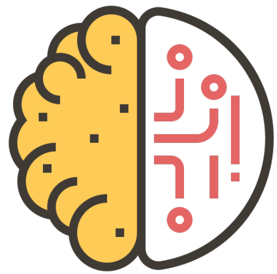

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Main problem: Counting • is very-limited and local. How to express:

- $\mathtt{Zeus}$ is the only $\mathit{KingOfGods}$?

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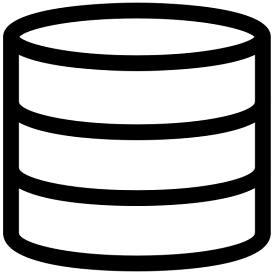

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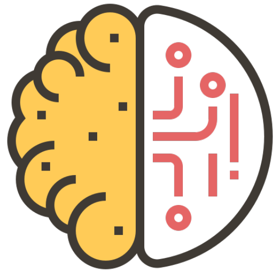

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Main problem: Counting • is very-limited and local. How to express:

- Zeus is the only *KingOfGods*? there are exactly 12 *Titans*?

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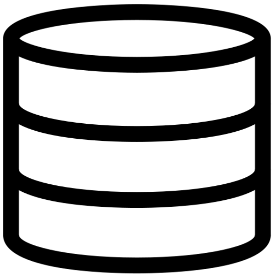

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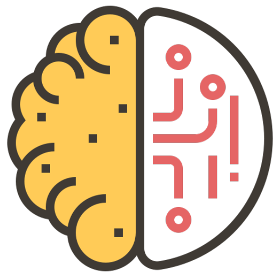

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Main problem: Counting • is very-limited and local. How to express:

- $\mathtt{Zeus}$ is the only *KingOfGods*? there are exactly 12 *Titans*?
- No more than 40% of $\mathtt{Zeus}$' children are *Male*?

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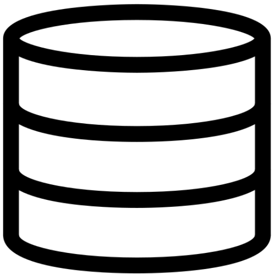

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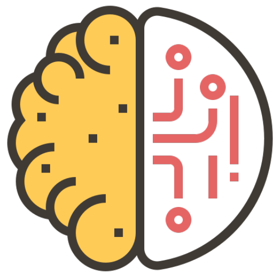

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Main problem: Counting is very-limited and local. How to express:

- $\mathtt{Zeus}$ is the only *KingOfGods*? there are exactly 12 *Titans*?
- No more than 40% of $\mathtt{Zeus}$' children are *Male*?
- Most of people stored in DB are $\mathtt{Zeus}$' children?

QFBAPA

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Quantifier-Free Boolean Algebra with Presburger Arithmetics

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counting toolkit

QFBAPA

Quantifier-Free Boolean Algebra with Presburger Arithmetics

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Step 1. Boolean algebra of sets.

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- Set variables: S, T, ...

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Example: $\emptyset \cup (S^c \cap (T \cup R)^c)$

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- Integer constants: $.., -1, 0, 1, 2, ..$
- Set cardinalities constants: $|S|$

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- Integer constants: $.., -1, 0, 1, 2, ..$
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QFBAPA

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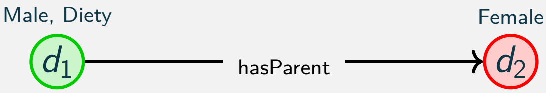

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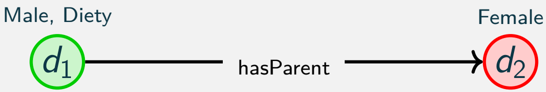

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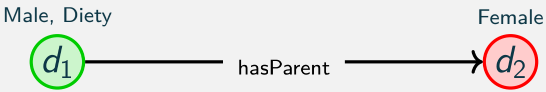
$\mathcal{I} =$ 

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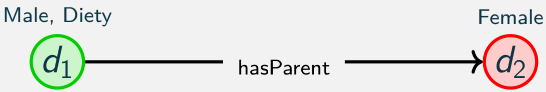
$\mathcal{I} =$ 

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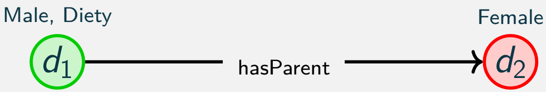
$\mathcal{I} =$ 

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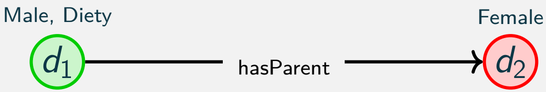

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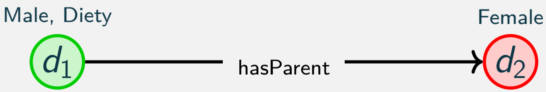

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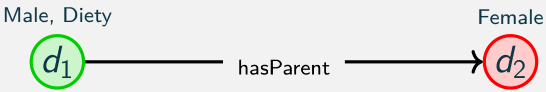
$\mathcal{I} =$ 

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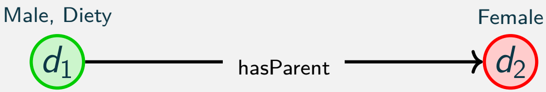
$\mathcal{I} =$ 

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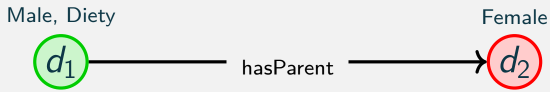
$\mathcal{I} =$ 

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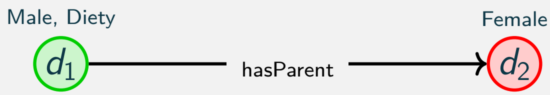

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