Guarded Fragments Meet Dynamic Logic
The Story of Regular Guards

KR 17.11.25, Melbourne, Austria

Bartosz Jan Bednarczyk `bartek@cs.uni.wroc.pl`

(Joint work with Emanuel Kieroński)

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Proving Useless Complexity Results For Random Logics

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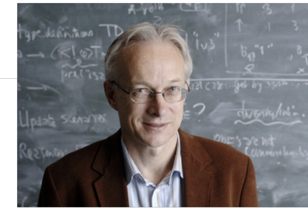

MODAL LANGUAGES AND BOUNDED FRAGMENTS OF PREDICATE LOGIC

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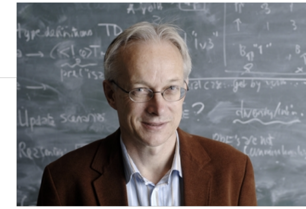

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Example 1. Some artist admires only beekeepers

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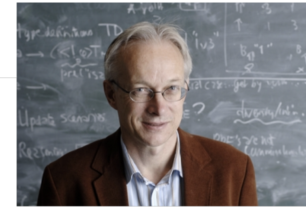

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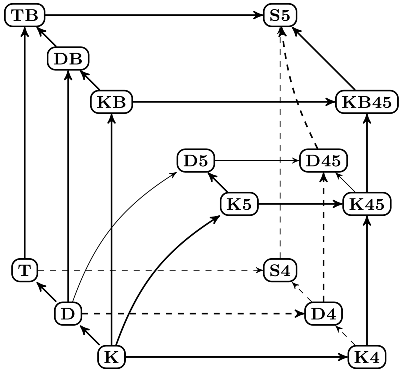

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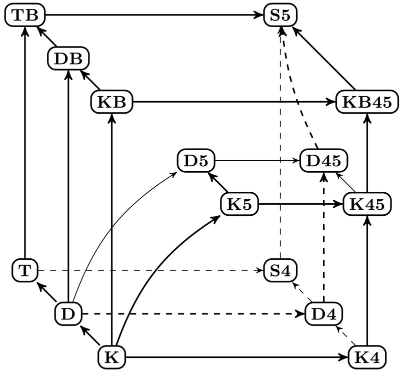

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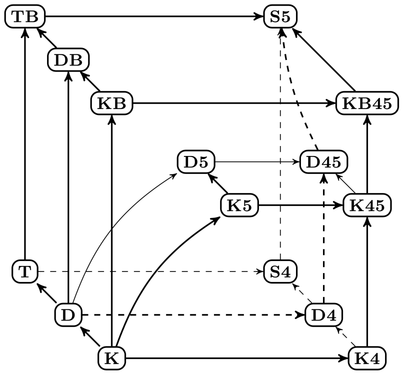

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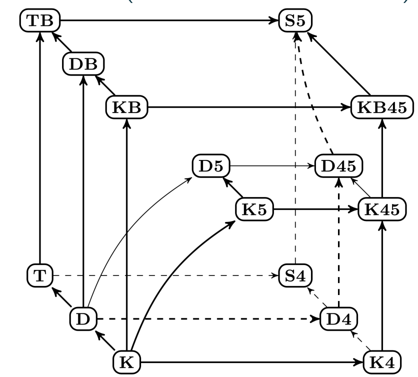

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The Two-Variable Guarded Fragment with Transitive Relations

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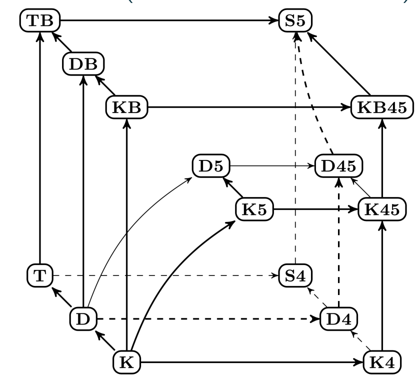

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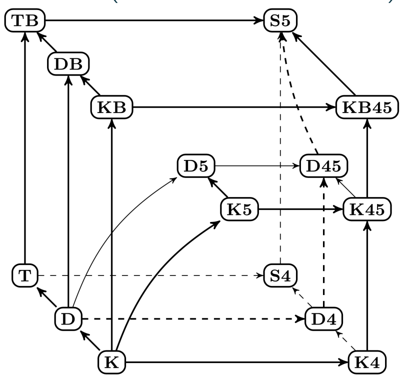

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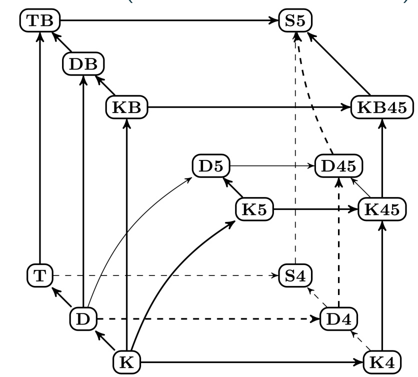

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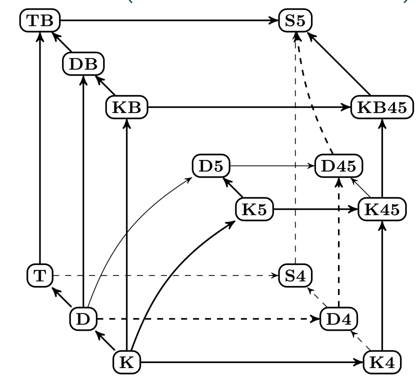

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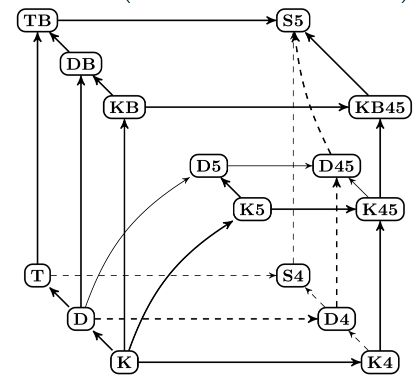

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ICPDL = ALC + $\exists(\pi \cap \ldots \cap \pi').\varphi$

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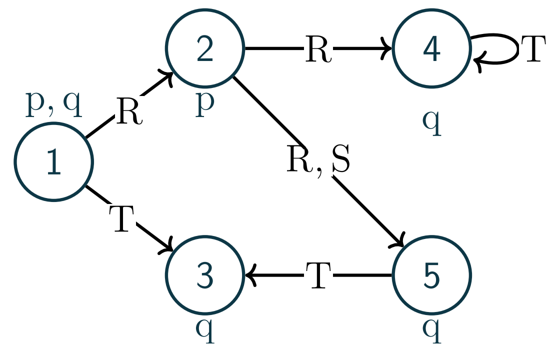

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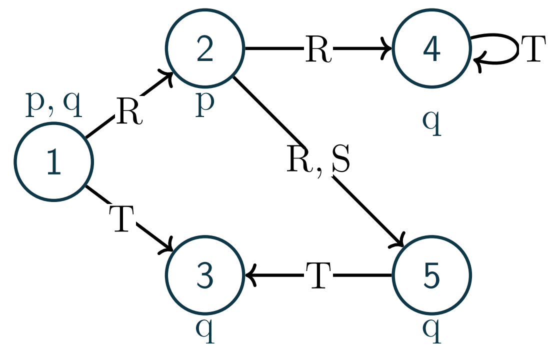

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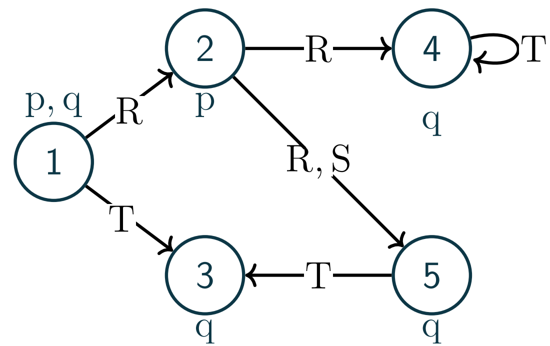

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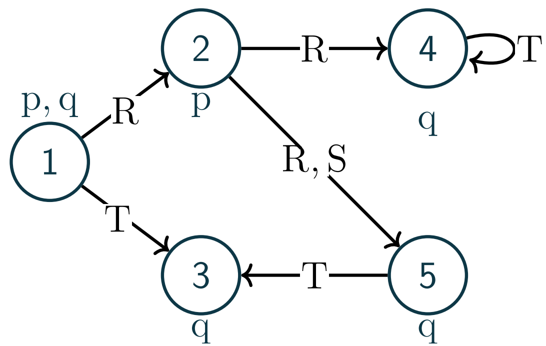

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Motivation IV: Provide a more high-level proof capturing many variants of GF in a uniform way.

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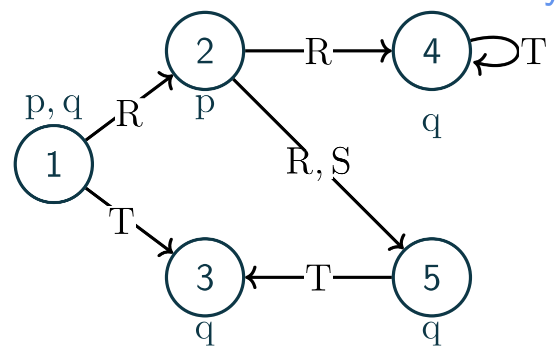

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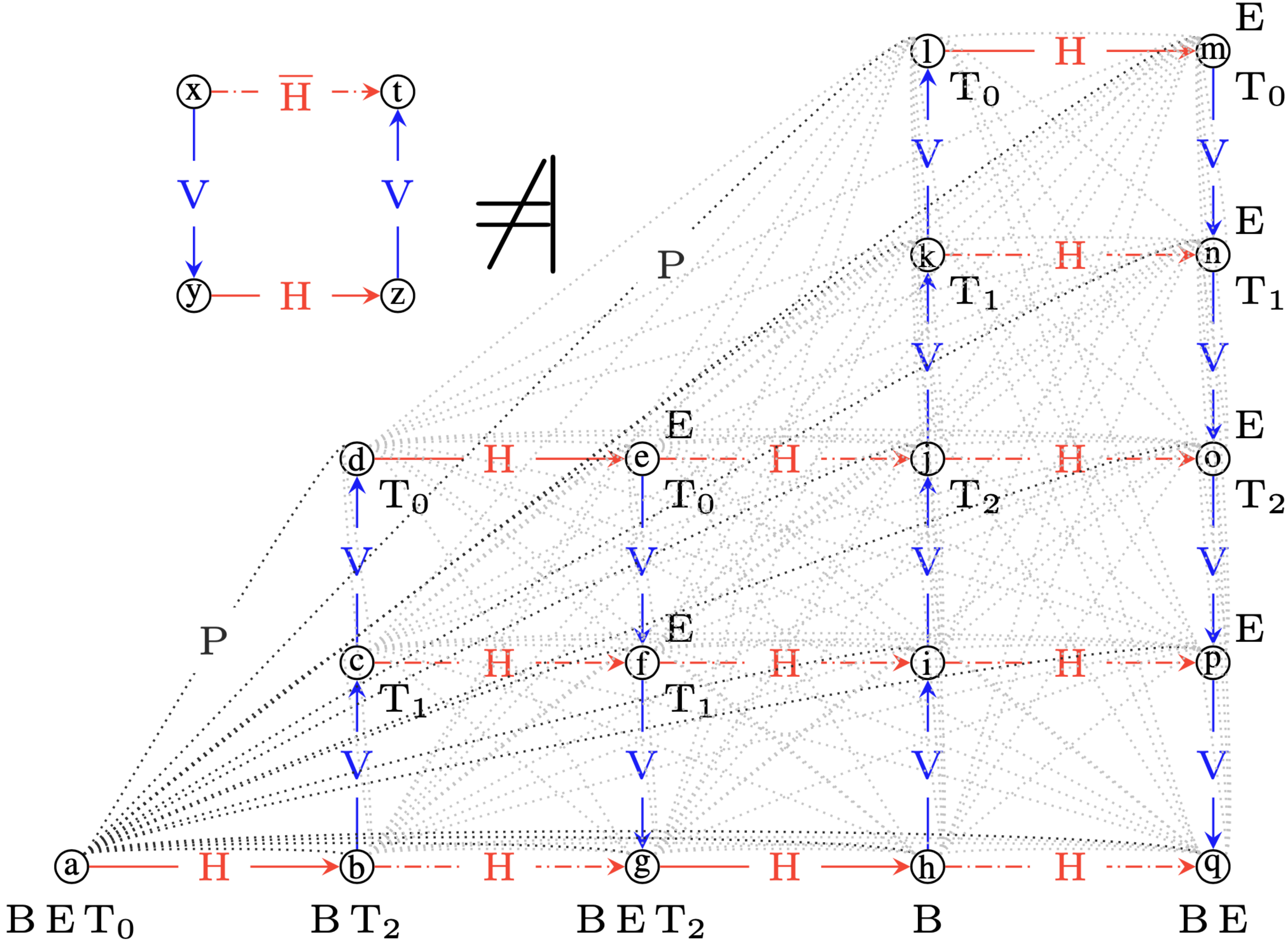
Significantly improves prior results by Gottlob&Pieris&Tendera from ICALP 2013.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- 1-type over Σ = maximal satisfiable conjunction of Σ-literals involving x_1. $(\boldsymbol{\alpha}_\Sigma)$

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Usually we rephrase satisfiaction of φ in terms of realizable types.

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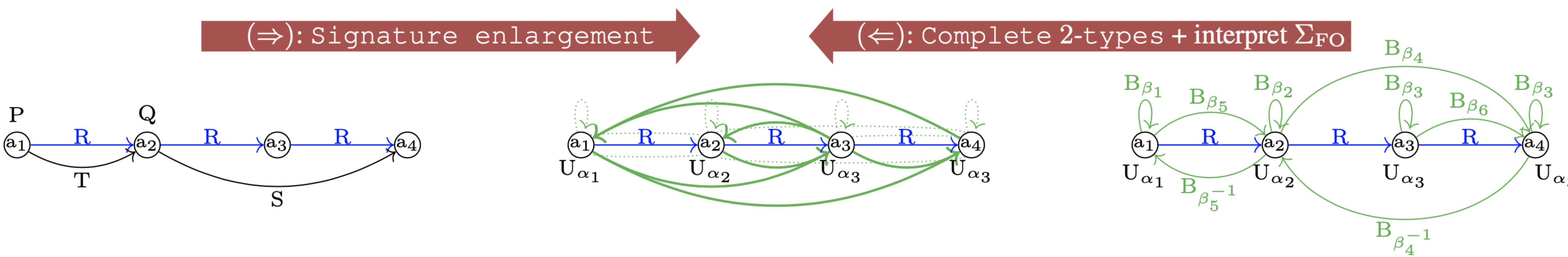

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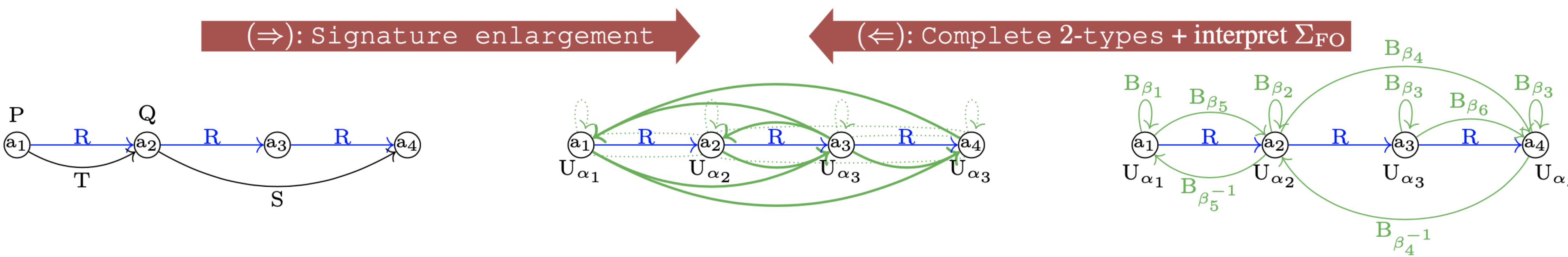

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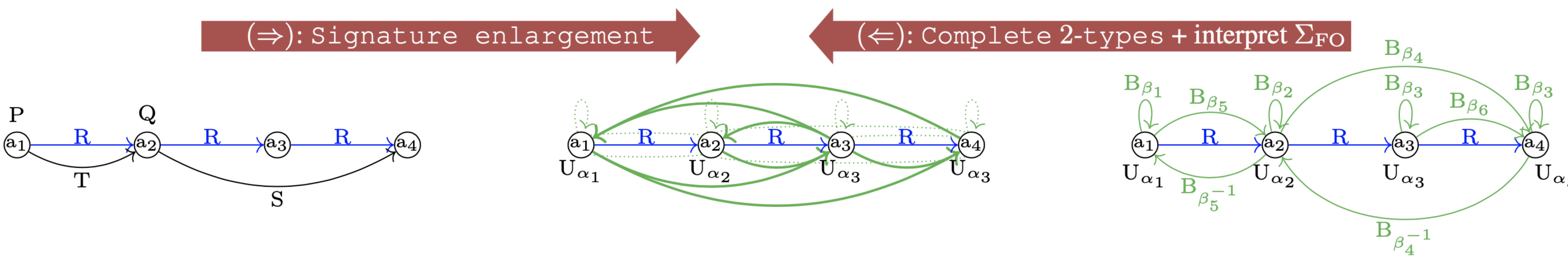

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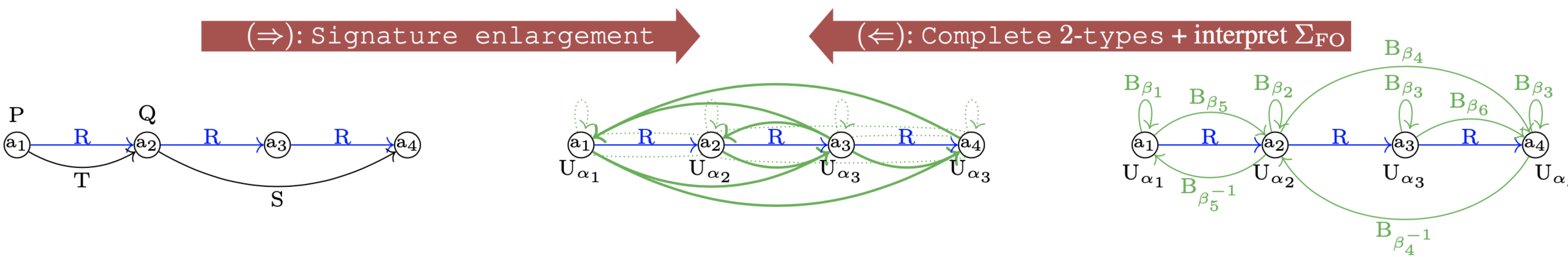

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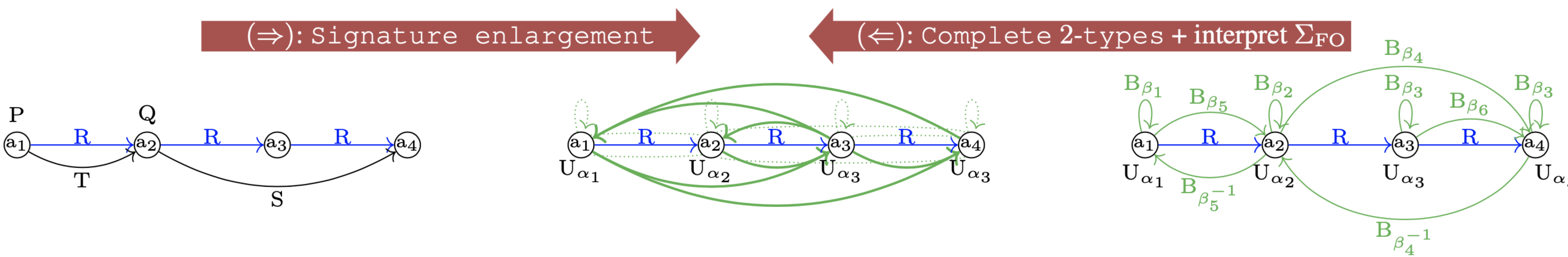

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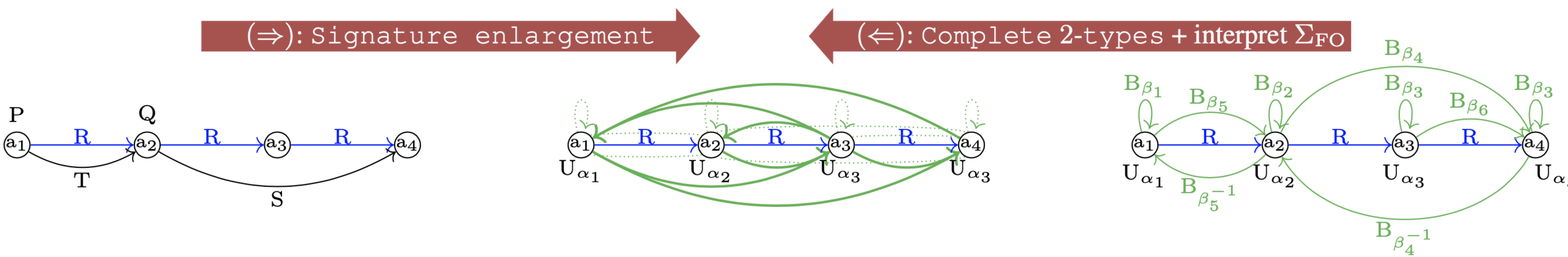

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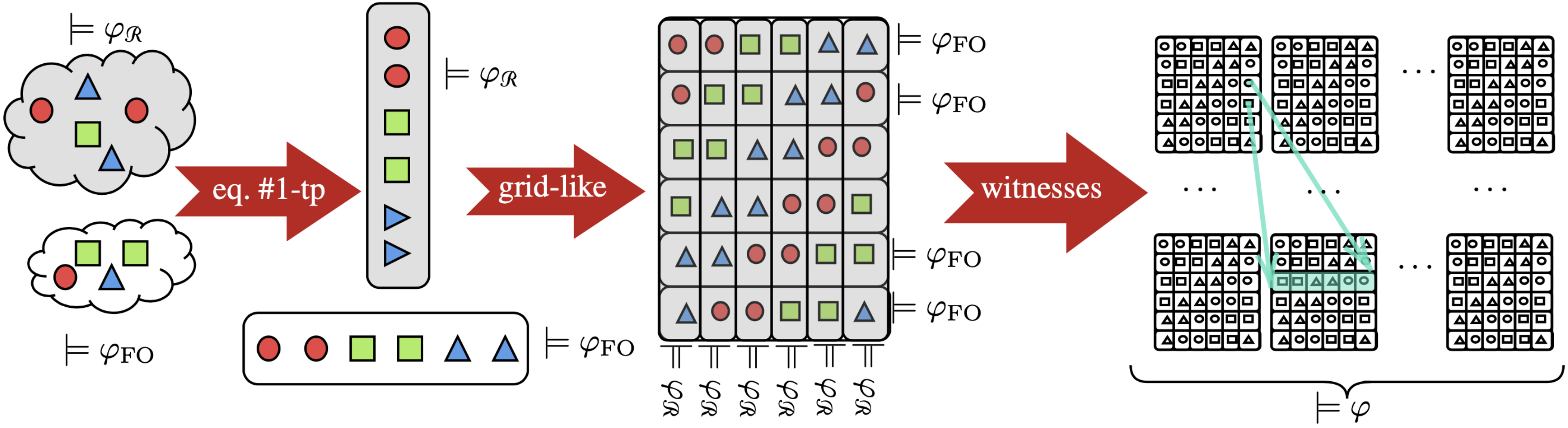
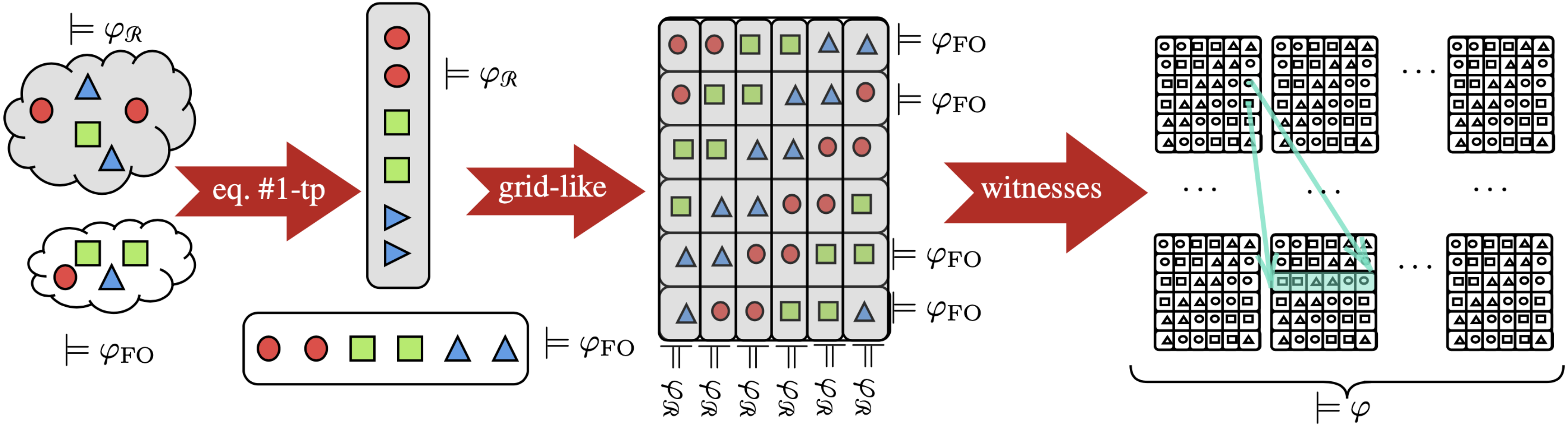


Correctness Proof: The Fusion



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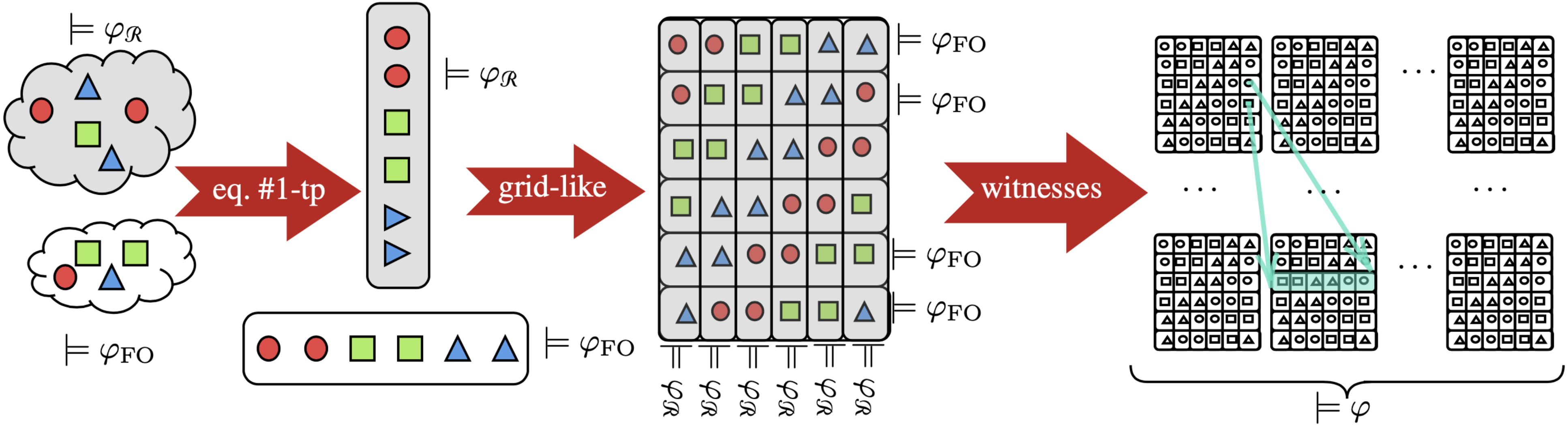

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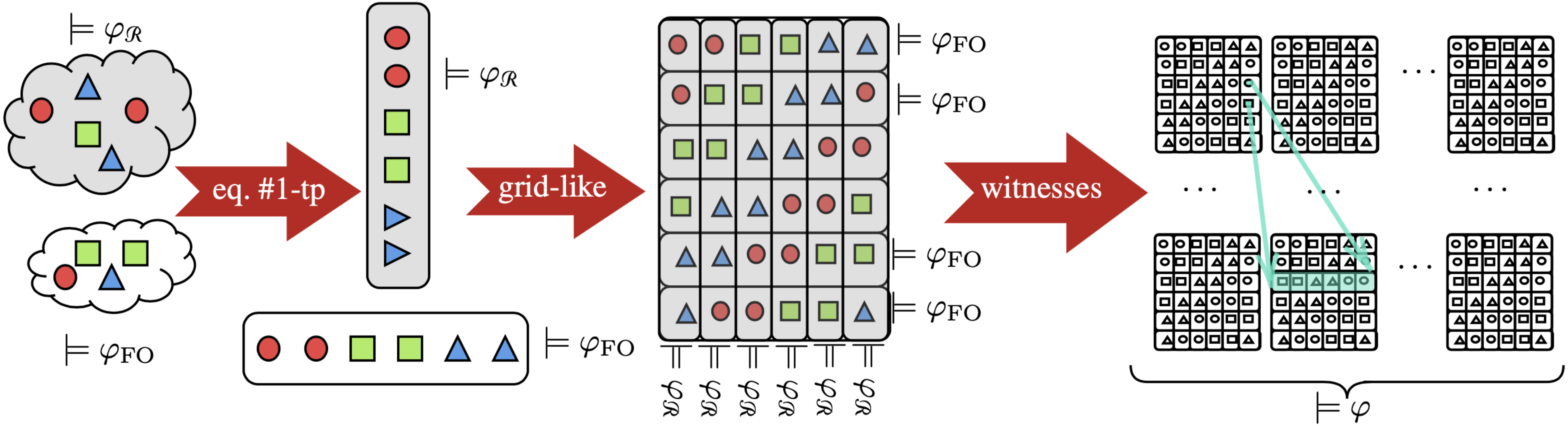

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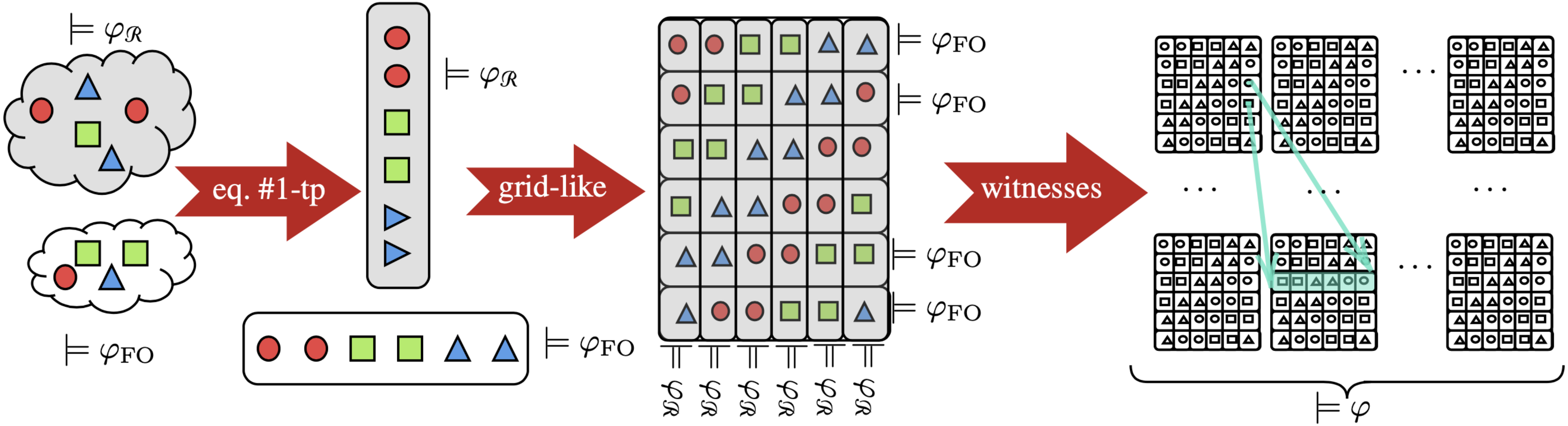

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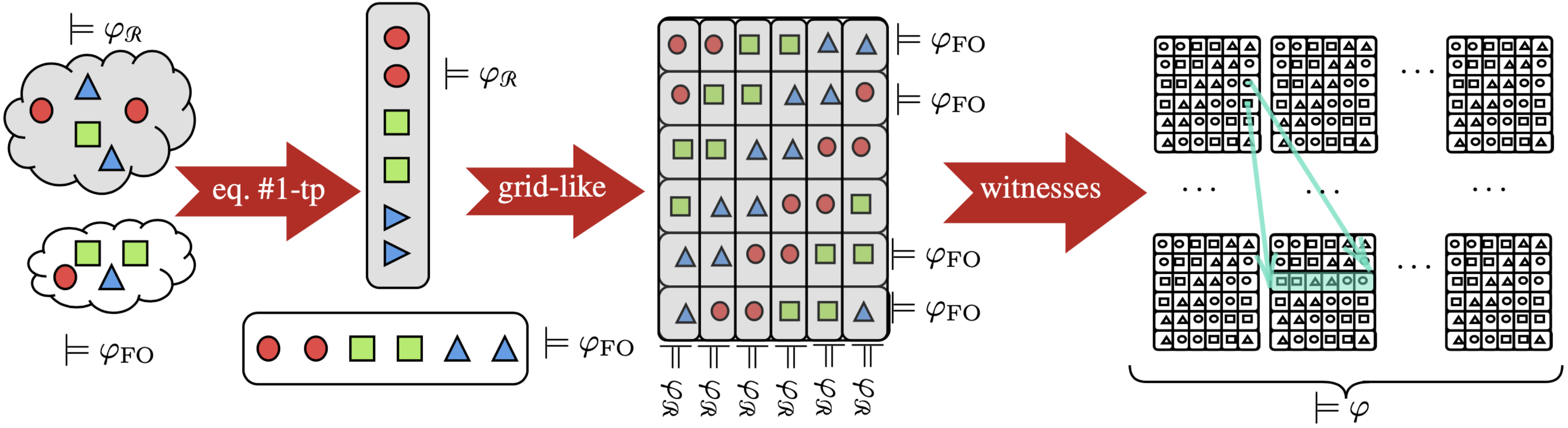

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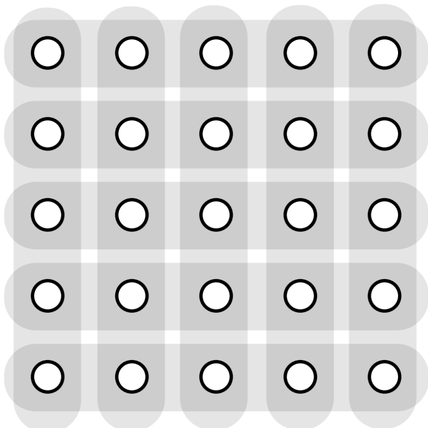

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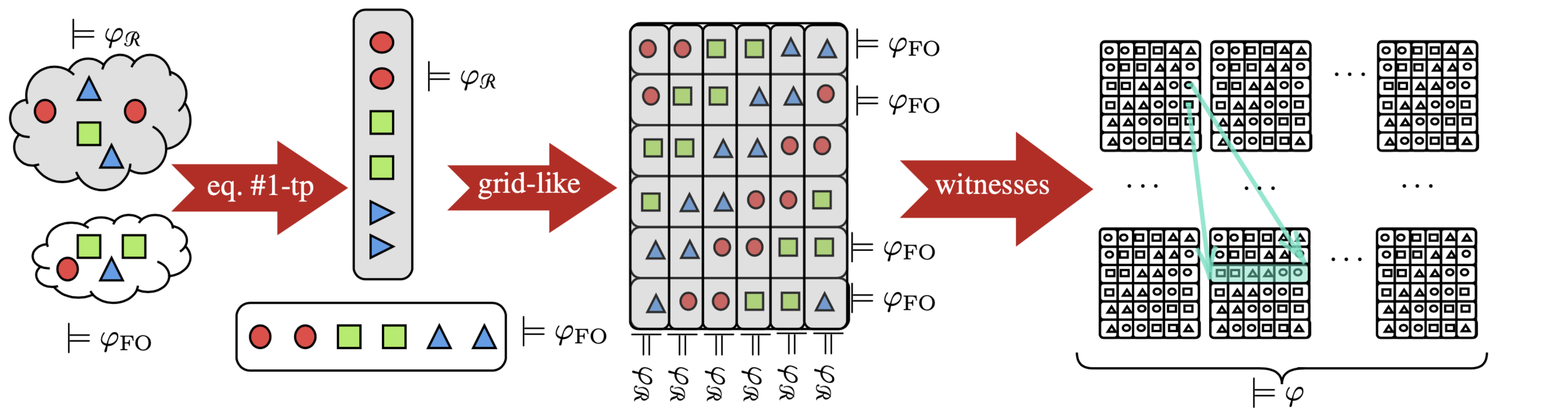

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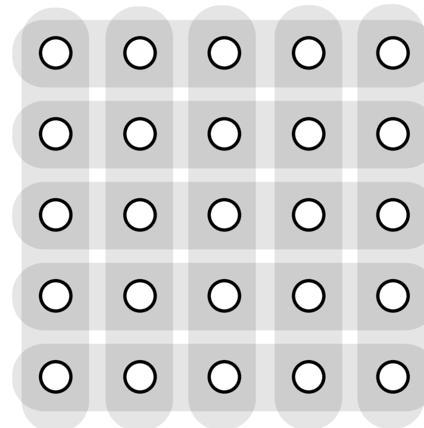

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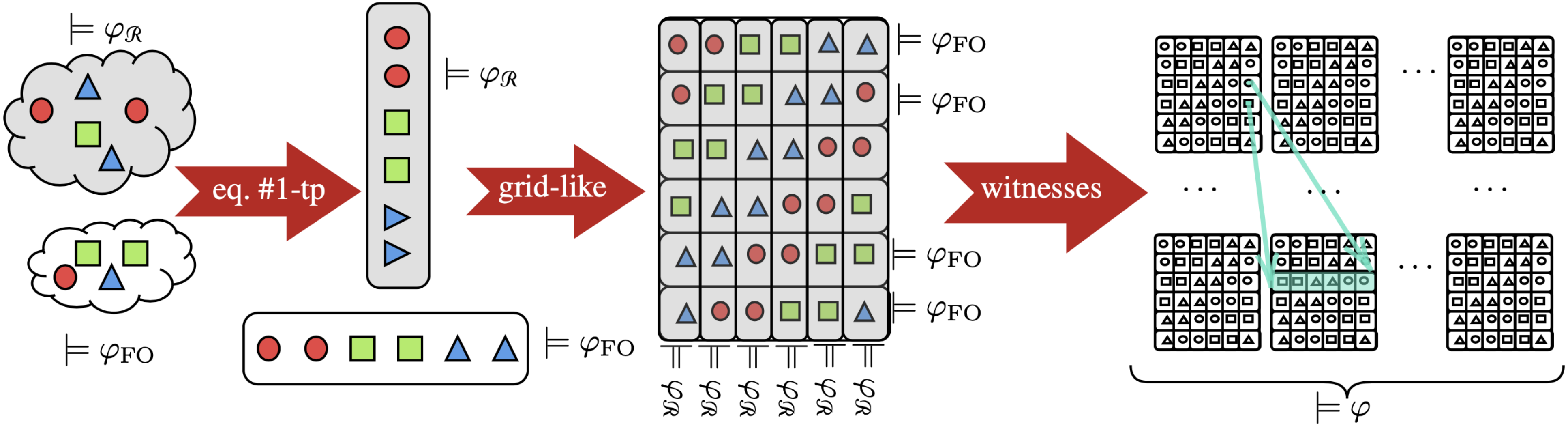
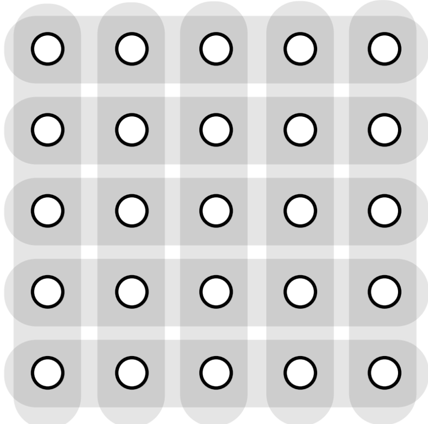
- $\mathfrak{C}$ is almost a model of φ, but some elements may miss witnesses for $\exists^{\mathsf{FO}}$-conjuncts.



Correctness Proof: The Fusion



- Take models $\mathfrak{A} \models \varphi_{\mathcal{R}}$ and $\mathfrak{A} \models \varphi_{\mathsf{GF}}$.
- Equilize their domains by Löwenheim Skolem (w.l.o.g. to $\aleph_0$).
- We next ensure that for each 1-type α in $\boldsymbol{\alpha}_{\varphi}$:

  $\#\alpha$ in $\mathfrak{A}$ is equal to $\#\alpha$ in $\mathfrak{B}$.
- Form a two-dimensional grid $\mathfrak{C}$. Each row (column) isomorphic to $\mathfrak{A}$ ($\mathfrak{B}$).
- $\mathfrak{C}$ is almost a model of φ, but some elements may miss witnesses for $\exists^{\mathsf{FO}}$-conjuncts.
- Provide them with a circular-witnessing-scheme á la Grädel & Kolaitis & Vardi.

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FINGO$^{\mu}$

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